D-Math | Functional Analysis 1 | HS 2014
Prof. Manfred Einsiedler

Serie 6

1. **a)** Prove that a real Banach space $(B, \|\cdot\|)$ is a Hilbert space (that is, $\|\cdot\|$ is induced by some inner product on B) if and only if $\|\cdot\|$ satisfies the parallelogram identity.

b) Prove that the norm of a Hilbert space is strictly sub-additive.

c) Use the Fréchet-Riesz Representation Theorem to show that if H is a Hilbert space, then H^* is also a Hilbert space, and exhibit a natural isometric isomorphism between H and H^{**}.

2. Suppose that H_1 and H_2 are two Hilbert spaces and $T \in B(H_1, H_2)$. Show the following:

a) There is a unique $T^* \in B(H_2, H_1)$, called the adjoint of T, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in H_1$, $y \in H_2$.

b) $\|T^*\| = \|T\|$, $\|T^*T\| = \|T\|^2$, $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$, $(ST)^* = T^*S^*$, and $T^{**} = T$.

c) Let R and N denote range and kernel, respectively. Then $R(T)^\perp = N(T^*)$ and $N(T)^\perp = \overline{R(T^*)}$.

d) T is unitary, that is, T is invertible and an isometry, if and only if T is invertible and $T^{-1} = T^*$.

3. Let (X, μ, Σ) be a measure space. We define a measure preserving transformation to be a measurable map $T : X \to X$ such that for any $A \in \Sigma$, $\mu(T^{-1}(A)) = \mu(A)$. Such a transformation induces a transformation U_T on $L^2(X, \mu)$ given by

$$f \mapsto U_T(f) = f \circ T.$$

a) Show that U_T is a well-defined transformation from $L^2(X, \mu)$ to itself, and an isometry.

b) Assume T is invertible as a measure preserving transformation, that is, T^{-1} exists, is measurable and measure preserving. Show that U_T is unitary.

4. Let $V = l^1(\mathbb{N})$ and fix some sequence a_n taking values in $(0, 1)$ with $\lim_n a_n = 1$.

a) Show that the set $K = \{x \in l^1(N) : x_n \geq 0 \text{ and } \sum_n a_n x_n = 1\}$ is closed and convex.

b) Let $v_0 = 0$ and prove $\inf_{x \in K} \|x - v_0\| = 1$.

c) Show that there does not exist some $x \in K$ with $\|x - v_0\| = 1$. Conclude in particular that the unit ball of $l^1(N)$ cannot be compact.