Serie 7

1. Show that the closed unit ball of a Hilbert space H is compact if and only if $\dim H$ is finite.

2. Let U be an unitary operator, that is isometric and surjective, on a Hilbert space H. Let $I = \{v \in H \mid Uv = v\}$ be the subspace of invariant vectors.

a) Show that $\{Uw - w \mid w \in H\}$ is dense in $I^{\perp}$ and that I is closed.

b) Let P is the orthogonal projection onto I. Show that

$$\frac{1}{N}\sum_{n=1}^{N} U^n v \to Pv.$$

Remark: This is von Neumann's mean ergodic theorem and applies in particular for $H = L^2(X, \mu)$ and $Uf = f \circ T$ where T is an invertible measure preserving transformation on X in the sense that $\int f \circ T d\mu = \int f d\mu$ for all $f \in L^2(X, \mu)$.

3. Let $A^2(D)$ be the Bergman space for the open unit disc $D \subset \mathbb{C}$. In Serie 5 it was proven that $A^2(D)$ is a Banach space.

a) Note that $A^2(D)$ is a separable Hilbert space. Prove that there is a *reproducing kernel* $K : D \times D \to \mathbb{C}$ such that for any $f \in A^2(D)$

$$f(z) = \int_D K(z, \zeta) f(\zeta) d\zeta.$$

b) Show that K is conjugate symmetric, that is, $K(z, \zeta) = \overline{K(\zeta, z)}$. Prove that K is the unique function that is reproducing, conjugate symmetric and $K(z, \cdot) \in A^2$

c) Show that

$$P : f \mapsto \int_D K(\cdot, \zeta) f(\zeta) d\zeta$$

is a Hilbert space orthogonal projection of $L^2(D, dz)$ onto $A^2(D)$.

d) Let $\{\varphi_i\}$ be an orthonormal basis of $A^2(D)$. Show that $\sum_{i=1}^{\infty} \varphi_i(z)\overline{\varphi_i(\zeta)}$ sums uniformly to $K(z, \zeta)$ on compact subsets of $D \times D$. In particular, $K(\cdot, \zeta) = \sum_{i=1}^{\infty} \varphi_i(\cdot)\overline{\varphi_i(\zeta)}$ in $A^2(D)$.

e) Show that $\{z^j\}_{j=0}^{\infty}$ is an orthogonal basis of $A^2(D)$ and calculate K explicitly.

4. Let B be a Banach space. A subspace $U \subset B$ is called complemented if there is a subspace $V \subset B$ such that the linear map

$$\begin{array}{ccc} U \oplus V & \longrightarrow & B \\ (u, v) & \longmapsto & u + v \end{array}$$

defines an isomorphism of Banach spaces.

Please turn over!

a) Show that U is complemented if and only if there is a continuous linear map $P : B \to B$ with $P^2 = P$ and $\operatorname{im} P = U$.

b) Show that a complemented space must be closed and $V \simeq B/U$ as Banach spaces. Recall that $\|v\|_V = \inf_{u \in U} \|v + u\|_B$.

c) The converse is not always true: We claim that the closed subspace $c_0 = \{x \in l^\infty : \lim x_n = 0\}$ has no complement in l^∞. Note that $(l^\infty)^*$ contains a countable subset L for which holds that if $l(x) = 0$ for all $l \in L$ then $x = 0$ and that this property is preserved under isomorphisms and taking subspaces. Show that $V \simeq l^\infty/c_0$ cannot have this property to give a contradiction to $V \oplus c_0 = l^\infty$. You may proceed as follows:

1. By enumerating $\mathbb{Q}$, construct for each $i \in I = \mathbb{R} \setminus \mathbb{Q}$ a 0-1-valued sequence $x_i \in l^\infty$ with infinite support $\operatorname{supp} x_i = \{n : (x_i)_n \neq 0\}$ for which which $\operatorname{supp} x_i \cap \operatorname{supp} x_j$ is finite for all $i \neq j$.
2. Show that for any $J \subset I$ finite, and any $b_i \in \mathbb{C}$ of absolute value one it holds $\| \sum_{i \in J} b_i x_i \|_V = 1$.
3. Use the last part to show that for any $f \in V^*$ and any $n \in \mathbb{N}$ the set $\{i \in I : |f(x_i)| > 1/n\}$ is finite.
4. Conclude that for any countable subset L of V^* there must exist $i \in I$ for which $l(x_i) = 0$ for all $l \in L$.