Serie 8

1. Let (X, μ) be a measure space and consider $(L^p(X), \|\cdot\|_p)$.

a) Let $p \in [2, \infty)$ and show that $(a^p + b^p)^2 \le (a^2 + b^2)^p$. Note that you can w.l.o.g. assume $a = 1$ and $b < 1$ and taking the derivative may simplifies the problem.

b) Use Jensen inequality to prove the following analogue of the parallelogram identity (which holds only for $p = 2$)

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \le \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p$$

for $p \ge 2$. and deduce that $L^p(X)$ is uniform convex for $p \in [2, \infty)$.

c) Show that $L^\infty(X)$ is in general not uniformly convex.

d) Show that $L^1(X)$ is in general not uniformly convex.

2. For $f \in L^1(\mathbb{T})$ define the nth Fourier coefficient by

$$\hat{f}_n := \int_0^1 f(t)\chi_n(-t)\mathrm{d}t.$$

Show that $\hat{f} = (\hat{f}_n) \in c_0(\mathbb{N}) \subset l^\infty(\mathbb{N})$ and $\|\hat{f}\|_{l^\infty} \le \|f\|_{L^1}$.

3. Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be closed subspaces of a Hilbert space $\mathcal{H}$ that are mutually orthogonal (possibly $\mathcal{H}_n = \{0\}$ for n large). Define the Hilbert orthogonal sum $\bigoplus \mathcal{H}_n$ to be the set

$$\bigoplus_n \mathcal{H}_n := \left\{ \sum_{n=0}^\infty v_n \in \mathcal{H} : v_n \in \mathcal{H}_n \right\}.$$

Prove that

$$\overline{\operatorname{span} \cup_n \mathcal{H}_n} = \bigoplus_n \mathcal{H}_n$$

and that for any converging sum $\sum_{n=0}^\infty v_n \in \bigoplus_n \mathcal{H}_n$ one has $\sum_{n=0}^\infty \|v_n\|^2 < \infty$.

4. Consider the action of $\mathrm{SO}(2) \simeq \mathbb{T}$ on $\mathbb{R}^2$ by rotation with $k_\vartheta = \begin{pmatrix} \cos 2\pi\vartheta & -\sin 2\pi\vartheta \\ \sin 2\pi\vartheta & \cos 2\pi\vartheta \end{pmatrix}$ and denote the corresponding action on $L^2(\mathbb{R}^2)$ by

$$(k_\vartheta f)(x, y) = f((x, y)k_\vartheta).$$

Let $f \in L^2(\mathbb{R}^2)$ and define $P_n f = f_n$ by

$$f_n(x, y) = \int_{\mathbb{T}} f((x, y)k_\vartheta)\mathrm{e}^{-2\pi i n\vartheta}\mathrm{d}\vartheta.$$

Please turn over!

a) Show that $P_n : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is continuous and that f_n is an eigenfunction for the action of $\mathrm{SO}(2)$ in the sense that there exists a function $\vartheta \mapsto \chi_n(\vartheta)$ such that $k_\vartheta f_n = \chi_n(\vartheta) f_n$.

b) Let $\mathcal{H}_n$ denote the subspace of $L^2(\mathbb{R}^2)$ consisting of functions for which $k_\vartheta f = \chi_n(\vartheta) f$. Prove that $\mathcal{H}_n$ are closed and mutually disjoint. Show that P_n is the orthogonal projection to $\mathcal{H}_n$ and

$$L^2(\mathbb{R}^2) \simeq \bigoplus \mathcal{H}_n$$

so that in particular $\sum_n f_n = f$ in L^2.

c) Show that if $f \in C_c^\infty(\mathbb{R}^2)$ then $\sum_n f_n$ converges to f uniformly on $\mathbb{R}^2$ by upgrading the corresponding result (Theorem 3.11) on $\mathbb{T}$.