Serie 9

1. Define $C^{(4)}(\mathbb{T}^2)$ to be the space of continuous function f for which the partial derivatives $\partial_x^4 f$ and $\partial_y^4 f$ exist and are continuous. Prove that $C^{(4)}(\mathbb{T}^2)$ is contained in

$$C^2(\mathbb{T}^2) = \{f \in C(\mathbb{T}^2) : \partial_\alpha f \text{ exists and is continuous for any } |\alpha| \leq 2 \}$$

where the ∂_α denotes a polynomial in the partial derivatives ∂_x, ∂_y of degree $|\alpha|$. Do this by comparing the Fourier series of $\partial_x^i f$, $\partial_y^j f$ and $\partial_x \partial_y f$.

2. Show that for a locally compact group G endowed with a Haar measure m_G the space $L^1(G, m_G)$ is a Banach-algebra where multiplication is given by convolution,

$$f * g(x) = \int_G f(xy^{-1}) g(y) dm_G(y).$$

When is it commutative? When does it contain a unit? Give a continuous injective homomorphism of the algebra $l^1(\mathbb{Z}) = L^1(\mathbb{Z}, \sum_{n\in\mathbb{Z}} \delta_n)$ to $C(\mathbb{T})$ where multiplication is defined by pointwise multiplication.

3. Consider the two subspaces $C_D = \{f \in C([0,1]) : f(0) = f(1) = 0\}$ and $C_N = \{f \in C^1([0,1]) : f'(0) = f'(1) = 0\}$.

a) Prove that $\{\sin \pi n x\}_{n>0}$ respectively $\{\cos \pi n x\}_{n\geq 0}$ is a Hilbertspace basis of the odd respectively even functions of $L^2(\mathbb{T})$.

b) Identify C_D and C_N with the subspace of odd functions of $C(\mathbb{T})$ respectively even function of $C^1(\mathbb{T})$.

4. Consider the family of translation operators $U_y : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ defined by $U_y f(x) = f(x+y)$ on the real line for $y \in \mathbb{R}$. We will analyze various continuity properties of this family in dependence of $p \in [1, \infty]$.

a) First show that for any fixed $y \in \mathbb{R}$ the operator $U_y : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ defines a bounded linear operator on $L^p(\mathbb{R})$ for any $p \in [1,\infty]$. What is its norm?

b) Fix $f \in L^p(\mathbb{R})$ and consider the map $\mathbb{R} \to L^p(\mathbb{R})$, $y \mapsto U_y(f)$. For which p is this map continuous?

c) Finally, prove that U_y is not continuous in the uniform operator topology, i.e. show that $y \mapsto U_y$ seen as a map from $\mathbb{R}$ to the space of bounded linear operators on $L^p(\mathbb{R})$ equipped with the usually operator norm is not continuous for any p!