Solution 1

1. Assume that k and m are chosen such that

$$k10^m \leq 2^n < (k+1)10^m,$$

i.e. that $k = l_n$. Taking the logarithm to base ten on both sides, gives $n\log_{10} 2 - m \in [\log_{10} k, \log_{10}(k+1))$. Since $\log_{10} k < 1$, m is such that $n\log_{10} 2 - m = n\log_{10} 2 \mod 1$. This implies that the number of occurrences of k as first digits in the sequence 2^n is the same as the number of elements $n\log_{10} 2 \mod 1$ in the interval $[\log_{10} k, \log_{10}(k+1))$. But by irrationality of $\log_{10} 2$ we know from example 1.9 in the script that

$$\frac{1}{N}|\{n\log_{10} 2 \mod 1 : n < N\}| \to |[\log_{10} k, \log_{10}(k+1))| = \log_{10}\frac{k+1}{k}.$$

2. By the Stone-Weierstrass Theorem and the usual 3ε-argument (see implication $4) \Rightarrow 2)$ chapter 1.2 for exactly this step) it suffices to prove that

$$\frac{1}{T}\int_0^T e(n\cdot(x_0+tv))dt \to \int_{\mathbb{T}^2} e(n\cdot x)dx$$

where $e(n\cdot x)$ is the n'th character $e^{2\pi i(x_1n_1+x_2n_2)}$. For $n=(0,0)$ this is trivial, so assume $n\in\mathbb{Z}^2_{\neq 0}$. First note that then $\int_{\mathbb{T}^2} e(n\cdot x) = 0$ since the exponential again integrates to a periodic function. For the left-hand side we integrate as well,

$$\frac{1}{T}\int_0^T e(n\cdot(x_0+tv)) = \frac{e(n\cdot x_0)}{T}\int_0^T e(n\cdot tv)dt = \frac{e(n\cdot x_0)}{T}\frac{1}{2\pi i(\alpha n_1+\beta n_2)}\left[e(n\cdot tv)\right]_0^T \to 0$$

as $T\to\infty$ since $e(\cdot)$ is bounded. Of course, we have to ensure that $\alpha n_1+\beta n_2$ does not vanish for any $n\in\mathbb{Z}^2_{\neq 0}$ which is equivalent to say that α and β are linearly independent. On the other hand, if $\alpha = \frac{n}{m}\beta$ then we can easily see that the character associated to $(n,-m)$, integrated over this rational line does not vanish.

3. **a)** To show that $p(fg'-f'g)$ is constant, we prove that its derivative $p'(fg'-f'g)+p(fg''-f''g)$ vanishes. But this term equals $L(f)g-L(g)f = (p'f'+pf''+qf)g-(p'g'+pg''+qg)f$ which is by assumption on f and g is zero. If $p>0$ then $p(fg'-f'g)\neq 0$. Indeed, by assumption that f and g are a fundamental solution, $f(a)g'(a)-f'(a)g(a)\neq 0$ so that $(f(a),f'(a))$ and $(g(a),g'(a))$ can span the two-dimension space of boundary conditions at a. If the fundamental solutions satisfy some regularity, i.e. are C^1, then $(fg'-f'g)\neq 0$ on a neighborhood of a. Thus $p(fg'-f'g)$ can only vanish if p would vanish on a neighborhood on a, but then L does not define a differential operator on this neighborhood (only multiplication by some continuous function which is trivial to solve), so that one might restrict again to the domain on which $p>0$ after all.

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b) We proceed as in the script: We split the integral $\int_a^b$ into two parts $\int_a^s$ and $\int_s^b$ to use the explicit formula for G:

$$\frac{1}{c}\left(\int_a^b G(s,t)h(t)dt\right)' = \left(\int_a^s f(s)g(t)h(t)dt + \int_s^b f(t)g(s)h(t)dt\right)'$$

$$= f(s)g(s)h(s) + \int_a^s f'(s)g(t)h(t)dt - f(s)g(s)h(s) + \int_s^b f(t)g'(s)h(t)dt$$

$$= \int_a^s f'(s)g(t)h(t)dt + \int_s^b f(t)g'(s)h(t)dt$$

so that

$$\frac{1}{c}\left(\int_a^b G(s,t)h(t)dt\right)'' = f'(s)g(s)h(s)dt + \int_a^s f''(s)g(t)h(t) - f(s)g'(s)h(s) + \int_s^b f(t)g''(s)h(t)dt.$$

We then collect the corresponding integrals in $L\left(\int_a^b G(s,t)h(t)dt\right)$, which equals

$$c\int_a^s L(f)(s)g(t)h(t)dt + c\int_s^b f(t)L(g)(s)h(t)dt + cp(s)(f'(s)g(s)h(s) - f(s)g'(s)h(s)) = chp(f'g - fg')(s).$$

By part b), and choosing $c^{-1} = p(f'g - fg')(s)$ we get that indeed $L\left(\int_a^b G(s,t)h(t)dt\right) = h$. The boundary condition is immediately implied by those of f and g. This finishes one part of the equivalence. For the converse, that is uniqueness, assume that H is a different solution to the same inhomogeneous boundary problem. Then $H - K(h)$ is a solution to the homogeneous problem $L(u) = 0$ with vanishing boundary conditions. But as f and g are assumed to be a basis of solutions, $H - K(h)$ is a linear combination of f and g. But the boundary conditions of $Af + Bg$ vanish if and only if $A = B = 0$ which proves $H = K(h)$.

4. We first collect the two existence theorems we want to use from chapter 1.4.1 on the heat equation. Firstly, we have a solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = b \end{cases}$$

where $b : \partial\Omega \to \mathbb{R}$ is a (nice) function on the boundary. Furthermore, assume that for a (again nice) function $f : \Omega \to \mathbb{R}$ with vanishing continuation to the boundary $\partial\Omega$ can be decomposed into functions $f_n : \overline{\Omega} \to \mathbb{R}$,

$$\begin{cases} f = \sum f_n \\ \Delta f_n = \lambda_n f_n \text{with } \lambda_n < 0 \\ f_n|_{\partial\Omega} = 0. \end{cases}$$

We now seek a solution to

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times [0,\infty) \\ u|_{\partial\Omega\times\{t\}} = b & \text{for all } t > 0 \\ u|_{\Omega\times\{0\}} = f \end{cases}$$

Denote by u_0 the solution to the Dirichlet boundary problem. We assume further that $f|_{\partial\Omega} = b$ in a sufficiently strong sense, such that the function $\tilde{f} = f - u_0$ is of the kind for which the decomposition in Laplace eigenfunctions holds. Using the ansatz of separating variables we can conclude that $u_n =$

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$f_n(x)e^{\lambda_n t}$ is a solution to the Heat equation with homogeneous boundary (meaning that $u_n|_{\partial\Omega} = 0$). Any finite linear combination of the u_n is also a solution, and so is $v = \lim_{n\to\infty} \sum u_n$ if for example $\sum u_n$ and $\sum \Delta_n u_n$ converge uniformly on Ω (this allows to interchange the two limits when differentiating and taking the infinite sum). At $t = 0$, $\sum u_n = \sum f_n = \tilde{f}$ by assumption. By linearity of the differential equation the function $u = u_0 + v$ is now easily seen to be a solution of the heat equation, $u_t = \Delta u$, and satisfies the boundary condition (in time and space), as $u|_{\partial\Omega\times\{t\}} = u_0|_{\partial\Omega} + v|_{\partial\Omega} = b + 0 = b$ and $u|_{\Omega\times\{0\}} = u_0|_{\Omega} + (f - u_0)|_{\Omega} = f$.