Solution 10

1. Consider the identity operator

$$K(x) = x = (x_1, x_2, \dots) \text{ on } l^2(\mathbb{N}).$$

Then the forgetful operator

$$K_n(x) = (x_1, \dots, x_n, 0, \dots)$$

is finite-dimensional, thus compact and clearly converges pointwise to K. However, K cannot be compact, since we have seen that the closed unit ball in infinite dimensionional Hilbert space is not compact.

2. a) We could apply Prop 4.10 to the integral kernel $(x, y) \mapsto \mathbb{1}_{[0,x]}(y)$ or use Lemma 4.7 applied to the K_n mentioned defined as follows. Approximate $K(x) = (x_1, x_2/2, x_3/3 \dots)$ by

$$K_n(x) = (x_1, \dots, x_n/n, 0, \dots).$$

Convergence is uniform,

$$\|Kx - K_n x\|^2 = \sum_{k=n+1}^{\infty} \frac{|x_k|^2}{k^2} \le \frac{1}{(n+1)^2} \sum_{k=n+1}^{\infty} |x_k|^2 \le \|x\|^2 \frac{1}{(n+1)^2},$$

so $\|K - K_n\|^2 \le \frac{1}{(n+1)^2} \to 0$ and Lemma 4.7 is applicable to deduce that K is compact. Note that we may extend K to $\mathbb{Z} - 0$ with the same compactness properties. Now, K relates to I via the Fourier transform F: $I(y \mapsto e^{2\pi i n y})(x) = \frac{1}{2\pi i n}(e^{2\pi i n x} - 1)$ for $n \neq 0$. Restrict to $L^2_0 = \{f \in L^2 : \int f = 0\}$, so that the Fourier coefficients $F(f)_n = a_n$ of $f \in L^2_0$ satisfy $a_0 = 0$. Let $(b_n)_{n\neq 0} = \frac{1}{2\pi i} K((a_n)_{n\neq 0})$ then

$$F(I(f))_m = F(I(\sum_{n\neq 0} a_n \chi_n))_m = (b_n)_{n\in\mathbb{Z}}$$

where $b_0 = -\sum_{n\neq 0} b_n = F(I(f))_0$. If P_0 denotes the projection from L^2 to L^2_0 and p_0 denotes the projection from l^2 to l^2_0 then we may write $K \circ F = p_0 \circ F \circ I \circ P_0$ which implies that K is compact if I is. On the other hand, assume that we know compactness of K and want to deduce it for I, and therefore have to take care of the constant functions that appear. Let $c \in L^2$ denote the constant function, P_c the orthogonal projection to the span of c (defined by integrating from 0 to 1) and p_c the projection to the 0 term in l^2. Then $I = I \circ P_0 + I \circ P_c$ is the decomposition of the bounded maps $(I \circ P_0, I \circ P_c)$ and the $+$-operation. Since P_c is compact (its image being finite dimensional), it suffices to show that $I \circ P_0$ is compact, F being an Hilbert space isomorphism, this is equivalent to show that $J = F \circ I \circ P_0$ is compact. But again we may write $J = p \circ J + p_c \circ J$. p_c is compact, and so is by assumption $p \circ J = K$ which implies that the composition (involving again the continous map $+$) is compact.

b) To calculate I^* denote χ_x the characteristic function $\mathbb{1}_{[0,x]}$, then

$$\langle f, I^*(g)\rangle = \langle I(f), g\rangle = \langle\langle \chi_x, f\rangle_{dt}, g\rangle_{dx} = \langle f, \langle \chi_x(t), g\rangle_{dx}\rangle_{dt}$$

by Fubini. Thus

$$I^*(g)(t) = \int_0^1 \chi_x(t) g(x) dx = \int_t^1 g(x) dx$$

which in the notation from above we can write as $I^* = P_c - I$.

Please turn over!

c) Let $\lambda \neq 0$ be a scalar. We have to show that $(\lambda \operatorname{Id} - I)$ is invertible. It suffices to show that the geometric series $\sum_{n=0}^{\infty} \frac{1}{\lambda^n} I^n$ converges in operator norm. To that end, we will show that $\sum \| \frac{1}{\lambda^n} I^n \|$ is finite. First let us understand the nth iteration of the operator n. We have

$$I^n(f)(x) = \int_0^x I^{n-1}(x_1) dx_1 = \cdots = \int_0^x \int_0^{x_1} \ldots \int_0^{x_{n-1}} f(x_n) dx_n \ldots dx_1.$$

Applying the Cauchy-Schwarz inequality to the inner integral, we make this iterated series of integrals independent of f:

$$|I^n(f)(x)| \leq \int_0^x \int_0^{x_1} \ldots \int_0^{x_{n-2}} \|f\|_2 \|\chi_{x_{n-1}}\|_2 dx_{n-1} \ldots dx_1.$$

Since $\|\chi_{x_{n-1}}\|_2 = x_{n-1}^{\frac{1}{2}}$ the iterated integral calculates to

$$x^{\frac{1}{2}+n-1} / \tfrac{1}{2}(\tfrac{1}{2}+1) \ldots (\tfrac{1}{2}+n-1) \leq \tfrac{1}{n!},$$

consequentially

$$\|I^n\| \leq \frac{1}{n!}.$$

which surely implies converges of the above mentioned sum that is bounded by $\exp \frac{1}{\lambda}$.

d) Since the spectrum is not empty, we conclude $\sigma(I) = \{0\}$. However, 0 cannot be an eigenvalue of I. The equality $\int_0^x f(t)dt = 0$ implies in particular that $f \in L_0^2$ and that $I(f) \in L_0^2$, and thus $0 = 2\pi i F(If) = KF(f)$ where F again denotes the Fourier transform. But if $f \neq 0$ then $F(f) \neq 0$. On the other hand, K is coordinatewise multiplication, and thus injective, which is a contradicting $KF(f) = 0$.

3. **a)** Recall that a mollifier J_ε is a sequence of bump functions that approximate the δ-function at some fixed point as $\varepsilon \to 0$. Here we approximate δ_0 the dirac measure at 0, in particular we may assume that the support of J_ε is sufficiently small (which holds as soon as ε is sufficiently small) such that J_ε defines a 1- periodic function. We first show that the first derivative of $f * J_\epsilon$ is $f_1 * J_\epsilon$. Assume for the moment that $D = \{x_0\}$ consists of just one point $x_0 \in [0, 1)$. Then

$$\begin{aligned}
\partial_x f * J_\epsilon(x) =& \partial_x \int_{\mathbb{R}} f(y) J_\epsilon(x-y) dy = \int_{\mathbb{R}} f(y) \partial_x J_\epsilon(x-y) dy \\
=& \int_0^{x_0} f(y) \partial_x J_\epsilon(x-y) dy + \int_{x_0}^1 f(y) \partial_x J_\epsilon(x-y) dy \\
=& \lim_{t \to x_0-} \int_0^t f(y) \partial_x J_\epsilon(x-y) dy + \lim_{t \to x_0+} \int_t^1 f(y) \partial_x J_\epsilon(x-y) dy \\
=& \lim_{t \to x_0-} \left(\int_0^t \partial_y f(y) J_\epsilon(x-y) dy - f(t) J_\epsilon(x-t) + f(0) J_\epsilon(x) \right) \\
&+ \lim_{t \to x_0+} \left(\int_t^1 \partial_y f(y) J_\epsilon(x-y) dy - f(1) J_\epsilon(x-1) + f(t) J_\epsilon(x-t) \right)
\end{aligned}$$

We could take the differential operator inside since $x \to f(y) J_\epsilon(x-y)$ is a differentiable map and (uniformly in x and y) bounded (since it is a product of continuous functions on a compact set) so one can apply dominated convergence. We then integrated by parts on both intervals where f is smooth. Now, by continuity and periodicity of f and J_ε the terms not involving $\partial_y f(y)$ cancel each other and we conclude

$$\partial_x f * J_\epsilon(x) = \int_{\mathbb{R}} f_1(y) J_\epsilon(x-y) dy.$$

See next page!

This argument of course extends to any other discrete set D. Since $*J_\epsilon$ maps $L^2(\mathbb{T})$ to itself we can infer that $f * J_\epsilon \in H^1(\mathbb{T})$. But the fact that

$$(f * J_\epsilon, f_1 * J_\epsilon) \to (f, f_1) \text{ in } L^2(\mathbb{T}) \times L^2(\mathbb{T}) \text{ as } \epsilon \to 0.$$

and $H^1(\mathbb{T}) \subset L^2(\mathbb{T}) \times L^2(\mathbb{T})$ is closed implies that $f * J_\epsilon$ converges actually in H^1, thus $f \in H^1(\mathbb{T})$.

Note that this argument also shows that happens if f is not continuous at x_0: Since $c = \lim_{t\to x_0+} f(t) - \lim_{t\to x_0-} f(t) > 0$, $\partial_x f * J_\epsilon$ will actually converge to $f_1 + c\delta_{x_0}$, since $J_\epsilon(\cdot - x_0) \to \delta_{x_0}$. This is understood weakly, that is, if we were to integrate $\partial_x f * J_\epsilon$ against a second L^1-function h, then $\int \partial_x f * J_\epsilon(x) h(x) \to \int f_1(x)h(x) + h(x_0)$.

b) We immediately deduce that if the derivatives (outside the singularities) up to order k define L^2 functions then $f \in H^k$. So let $f = |x|^\alpha$ on $(-\epsilon, \epsilon) \subset U$ and smooth everywhere else. In particular, by compactness of T, f (and all its derivatives) are square-integrable on the complement of $(-\epsilon, \epsilon)$ in $\mathbb{T}$. Our set of singularities is $D = \{0\}$ and by symmetry of $|x|^\alpha$, which is smooth on $(0, \epsilon)$ it suffices to check the existence of the integral

$$\int_0^\epsilon |\partial_x^k x^\alpha|^2$$

which is equal to a constant times

$$\lim_{\delta\to 0} x^{2(\alpha-k)+1}|_\delta^\epsilon$$

Here, we already assumed that $2(\alpha - k) \neq 1$ since the logarithm is not integrable at 0. This term is finite only if $\alpha + \frac{1}{2} > k$.

4. First some general remarks. We defined (Definition 3.48) $H^1(U)$ to be the closure of $C = \{(f, f') : f \in C^\infty(U)\}$ in $L^2(U) \oplus L^2(U) = V_1 \oplus V_2$ and may identify it as a subspace of $V = L^2(U)$, using the first projection $\iota : H^1(U) \to V_1 \to V$. But also the second projection is continuous and defines thus an operator $D : H^1 \to V_2 \to V$. When restricting to C, then $Df = f'$, so that we consider D as continuous extension of taking derivative in $H^1(U) \subset V$.

a) This is more or less example 3.57 including its proof, which implies that point evaluation l_0 at 0 is continuous in H^1 and so is taking the integral I is continuous (even compact by see exercise 2), we have an equality $\iota(f) = l_0 \circ \iota(f) + I \circ D(f)$ in C which (by linearity and continuity) remains true for the closure H^1.

On the other hand, we claim that any function $f \in C([0,1])$ of the form $f(x) = f(0) + I(g)(x)$ for some

Note that if $g \in L^2(U)$ then g is also absolutely integrable. This implies that $I(g)$ is absolutely continuous, and as such is almost everywhere differentiable with derivative equal to g (almost everywhere). Analogously to exercise 2, using mollifiers J_n as introduced in Serie 4 exercise 2 and we only sketch the argument. Approximate g in L^2 by $g_n = g * J_n$ (in Serie 4 we only proved convergence in L^1 but one can modify the argument to see that if g is in L^p then convergence is in L^p). It remains to verify that $I(g_n) \to I(g)$ in L^2. Since $I(h)(y) = \langle \chi_y, h\rangle$ for some function χ_y, it follows from $g_n \to g$ in L^2 that for fixed y, $I(g_n) \to I(g)$. To upgrate the pointwise convergence to L^2 convergence, apply dominated convergence.

For H_0^1 uses the additional equation $l_0(f) = l_1(f) = 0$.

b) See line (3.27) of the justification of example 3.57.

c) We omit the subscript 0 when restricting D to H_0^1. We have to solve for

$$\langle Df, g\rangle_{L^2} = \langle f, D^*g\rangle_{H_0^1} = \langle \iota(f), \iota((D^*g))\rangle_{L^2} + \langle Df, D(D^*g)\rangle_{L^2}.$$

Let $D^*g = (g_1, g_2)$ then $g_1 = I(g_2)$ by part a). From a) also as $I \circ D = \mathrm{id}$ (in H_0^1) so that the RHS reads

$$\langle f, Ig_2\rangle_{L^2} + \langle Df, g_2\rangle_{L^2} = \langle IDf, Ig_2\rangle_{L^2} + \langle Df, g_2\rangle_{L^2} = \langle Df, (I^*I - \mathrm{id})g_2\rangle_{L^2}$$

which implies that $D^* = (I^*I - \mathrm{id})^{-1}$. Similiarly to exercise 3c, one shows that

$$\|I^*I\| < 1$$

which implies invertibility of $I^*I - \mathrm{id}$.

d) Let now $\langle\cdot,\cdot\rangle_1$ denote the new scalar product on H_0^1, that is, we are taking the scalar product of the second factor only. Since $\iota : H_0^1 \to V_1 = L^2$, the adjoint $\iota^*(f)$ is a tuple of functions and $\iota \circ \iota^*$ is a map from L^2 to L^2. The adjoint satisfies by definition for $f \in H_0^1$ and $g \in L^2$,

$$\langle \iota(f), g\rangle_{L^2} = \langle f, \iota^*(g)\rangle_{H^1}.$$

The map $\iota\circ\iota^*$ is self-adjoint map on L^2 (for a bounded operator A, AA^* and A^*A are self-adjoint) and so is $-\iota \circ \iota^*$. We know that ι is compact, thus $\iota \circ \iota^*$ is a compact self-adjoint operator for which we have proven a spectral decomposition. If $\Delta \circ -\iota \circ \iota^*$ is the identity in L^2, then any eigenfunction of $-\iota \circ \iota^*$ is thus also one of Δ with reciprocal eigenvalue. To that end let $f \in L^2$ and $g \in C_c^2$ (a dense subset of H_0^1). We keep writing D for taking the derivative but in L^2 (so that $D \circ \iota$ corresponds to what previously was the projection onto the second coordinate in H_0^1 due to our identification with L^2)

Then by partial integration (and vanishing boundary terms of g)

$$\langle \Delta \circ -\iota \circ \iota^* f, g\rangle_{L^2} = \langle D(\iota \circ \iota^* f), Dg\rangle_{L^2} = \langle \iota^* f, g\rangle_1 = \langle f, g\rangle_{L^2}$$

e) The solutions for $\Delta f = \lambda f$ such that $f(0) = f(1) = 0$ can be found by making the ansatz using linear combination of fundamental solutions of the laplacian, $\sin\sqrt{\lambda}x$ and $\cos\sqrt{\lambda}x$, so that by the boundary conditions only sinus can appear. Again by the boundary conditions, $\lambda_n = (n\pi)^2$ for $n \in \mathbb{N}$ and the number of $n \in \mathbb{N}$ for which $\lambda_n \leq \Lambda$ is $\Lambda^{\frac{1}{2}}/\pi$.