Solution 11

1. The spectral theorem gives us an countable orthonormal basis $\{v_i\}$ of H consisting of eigenvectors of K, $Kv_i = \lambda_i v_i$, such that $\lambda_i \to 0$. By positivity of K we have $\lambda_i \geq 0$. Note that this implies that we can write $Kv = \sum_{i=1}^{\infty} \lambda_i \langle v, v_i \rangle v_i$ with convergence in H of the right hand side. This motivates to define

$$Sv = \sum \sqrt{\lambda_n} \langle v, v_n \rangle v_n.$$

S is bounded (by $\max \sqrt{\lambda_n}$) and satisfies $S \circ S = K$ for each eigenvector, so also for each finite sum and thus for all of H by density. We can also read of its definition that S is positive and all that remains to be shown is compactness. This follows from the more general statement (inverse of the spectral theorem) that any diagonal operator of the kind

$$Dv = \sum \mu_n \langle v, v_n \rangle v_n \text{ such that } \mu_n \to 0$$

is compact. This can be seen to be true, if we define

$$D_N = \sum_{n \leq N} \mu_n \langle v, v_n \rangle,$$

which is of finite-dimensional image, thus compact. D_N converges in operator norm to D because $D - D_N = \sum_{n>N} \mu_n \langle v, v_n \rangle$ is bounded by μ_N. Since by assumption $\mu_N \to 0$, it follows that $\|D - D_N\| \to 0$.

2. **a)** Let $\{T_x = F(x, \cdot) \in Y^*\}_{x \in \overline{B}_1^X} \subset Y^*$ be the family of bounded operators for which we want to apply the Banach-Steinhaus theorem. Here, x is restricted to the unit ball in X. By assumption we have $x \mapsto T_x(y) \in X^*$ for any fixed $y \in Y$:

$$\sup_{x \in \overline{B}_1^X} |T_x(y)| < \infty.$$

Banach-Steinhaus therefore tells us that $\{T_x\}$ is uniformly bounded:

$$M = \sup_{x \in \overline{B}_1^X} \|T_x\| < \infty.$$

We can conclude

$$\|F\| = \sup_{\|x\|, \|y\| \leq 1} |F(x,y)| \leq \sup_{\|x\|, \|y\| \leq 1} \left(\sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} |F(x,y)| \right) = M < \infty.$$

b) Define a family of operators on $\mathcal{H}$ by $l_y(x) = \langle x, Ay \rangle$ for $y \in \overline{B}_1^{\mathcal{H}}$. Note that by the Cauchy-Schwarz inequality we have

$$l_y(x) \leq \|x\| \|Ay\|,$$

that is, $l_y \in \mathcal{H}^*$ with $\|l_y\| \leq \|Ay\|$ (by Riesz, equality actually holds). Further, for fixed x we similarly find by assumption that $\langle x, Ay \rangle = \langle Ax, y \rangle$

$$\sup_{y \in \overline{B}_1^{\mathcal{H}}} |l_y(x)| \leq \sup_{y \in \overline{B}_1^{\mathcal{H}}} \|Ax\| \|y\| = \|Ax\| < \infty.$$

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This is the assumption of the Banach-Steinhaus theorem and we can conclude as before that

$$\sup_{\|y\|\leq 1} \sup_{\|x\|\leq 1} \langle Ax, y\rangle < \infty$$

and since

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{\|x\|\leq 1} \frac{1}{\|Ax\|}\langle Ax, Ax\rangle = \sup_{\|x\|\leq 1} \langle Ax, \frac{Ax}{\|Ax\|}\rangle.$$

The vector $\frac{Ax}{\|Ax\|}$ clearly has norm one, hence $\|A\|$ is bounded by $\sup_{\|y\|\leq 1} \sup_{\|x\|\leq 1}\langle Ax, y\rangle$.

3. a) The Hecke operator is obviously linear, since summing and point evaluation is. Self-adjointness will follow from the symmetry property $\sim$ of being a neighbor:

$$\langle T_p f, g\rangle = \sum_{v\in V} \sum_{v\sim w} f(w)\overline{g}(v).$$

A characteristic term of the sum $f(w)\overline{g}(v)$ appears precisely when $w \sim v$. But the last statement is symmetric in v and w and is thus also true for the sum

$$\sum_{v\in V} \sum_{v\sim w} f(v)\overline{g}(w) = \langle f, T_p g\rangle.$$

Continuity follows from an application of the closed graph theorem to any operator with the self-adjointness property that is defined everywhere since T_p is a finite sum in every point. We further have

$$\|T_p f(v)\|^2 = \sum_{v\in V} |\frac{1}{(p+1)} \sum_{v\sim w} f(w)|^2 \leq \frac{1}{(p+1)^2} \sum_{v\in W} \sum_{w\sim v\sim w'} f(w)\overline{f}(w').$$

A term $f(w)f(w')$ appears exactly once in the sum if $w \sim_2 w'$ are two-neighbors (from which there are $p(p+1)$ and $p+1$ times if $w = w'$. We get $(p+1)^2$ sums of the form $\sum f(w)\overline{f}(w')$ where no element appears twice, so that each of the sum can be written as $\langle f_1, f_2\rangle$ where f_1 and f_2 is some reordering of the sequence f. Now, one has to be careful and note, that we have to show that these f_1, f_2 are actual reorderings (bijections), but this follows from the fact that we can find an assignment of labeling the neighbors for each vertex, i.e. for each vertex v, we give the $p+1$ neighbors a name, $v_1, .., v_{p+1}$, in such a way that every vertex has a name (coming from its neighbors) so that every integer appears and this translate into saying, that the map 'shifting to the kth neighbor', $\tau \to \tau$, $v \to v_k$ becomes a bijection. Finally, by Cauchy-Schwartz we get the bound $\|T_p f(v)\|^2 \leq \frac{1}{(p+1)^2}(p+1)^2 = 1$ which, as we will see, is far from being the precise bound.

For those who are interested, this result also follows immediately from Riesz-Thurin which informally states that every operator that is defined on L^p and L^q with bound B on both spaces, is also bonded by B on all L^r, $p \leq r \leq q$. Here, $p = 1$, $q = \infty$.

b) Now as for the averaging operator S_n that takes any neighbors of distance less n into account, we first notice that we have 1 0-neighbor, $p+1$ 1-neighbor, $(p+1)p$ 2-neighbors and in generally $(p+1)p^{k-1}$ k-neighbors. It is further clear that for $\delta_{v_0} = e_0$ the sequence defined by

$$S_n e_0(v) = \sum_{w\sim_{\leq n} v} e_0(w)$$

has no mass for any $v \sim_{>n} v_0$ and is equal to one for any v with $v \sim_{\leq n} v_0$. The norm squared of $S_n e_0$ is then simply the number of vertices with mass one, which is $1+(p+1)+..+(p+1)p^{n-1} = 1 + \frac{p^n-1}{p-1}(p+1)$