# Solution 2

**1.** a) That $W$ is a subspace follows from the linearity of the integral. To see that $W$ is closed we just need to check the defining property, i.e. that the intregral over the left hand and right hand side of the interval vanishes. So, note that $-2\|f-f_n\|_\infty \leq \int_I (f(x)-f_n(x))d\mu(x) \leq 2\|f-f_n\|_\infty$ for any $I \subset [-1,1]$. Hence if $f_n \to f$ w.r.t. the supremum norm then $f \in W$.

b) Take $g(x) = x + \frac{1}{2}$ for $x < 0$ and $g(x) = x - \frac{1}{2}$ for $x \geq 0$. Then $|f(x) - g(x)| \equiv \frac{1}{2}$. To find a continuous $h$ that approximates $g$ we define

$$h(x) = \begin{cases} x + \frac{1}{2} + \epsilon & x < -\delta \\ mx & -\delta \leq x \leq \delta \\ x - \frac{1}{2} - \epsilon & x > \delta \end{cases}$$

where $m$ is chosen such that $h$ is continuous and $\delta$ is chosen such that $h \in W$. With that definition, $\|h - f\| = \frac{1}{2} + \epsilon$, which implies that $\|f\|_W \leq \frac{1}{2}$. The other inequality follows from the argument we give for c).

c) We claim that $h$ satisfies $x + h(x) \equiv \frac{1}{2}$ for positive $x$ if $h$ were chosen to archive that $\|f+h\| = \frac{1}{2}$. Notice that $\frac{1}{2} = \int_0^1 (x+h(x))dx \leq \int_0^1 |x+h(x)| < \frac{1}{2}$ if there would be some $x \in (0,1)$ in which $|x+h(x)| < \frac{1}{2}$ by continuity of this function. Similiar for negative $x$, hence $h = g$ is the found function in $b)$, thus discontinuous in $0$.

**2.** **a)** Let $d$ be a metric. Symmetry and strict positivity of $\frac{d}{d+1}$ immediately follows from the correspon-ding properties of $d$. The triangle equation needs a little calculation:

$$\frac{d(a,c)}{d(a,c)+1} \leq \frac{d(a,b)}{d(a,b)+1} + \frac{d(b,c)}{d(b,c)+1}$$

is equivalent to

$$0 \leq d(a,b)(d(a,c)+1)(d(b,c)+1) + d(b,c)(d(a,c)+1)(d(a,b)+1) - d(a,c)(d(b,c)+1)(d(a,b)+1)$$

The right hand site equals

$$d(a,b)d(b,c)d(a,c) + d(a,b)d(b,c) + d(b,c)d(a,b) + d(a,b) + d(b,c) - d(a,c)$$

which is positive by the triangle inequality for $d$. Alternatively, one notes that $x/1+x$ is monotone.

**b)** We first note that if $(V_n, d_n)$ is a sequence of metric spaces, where $d_n$ is bounded independently of $n$ (that is, there exists $C$ such that for all $n$, for all $x, y \in V_n$ one has $d_n(x,y) < C$) then $\sum 2^{-n} d_n$ defines a metric on the (possibly infinite) product space $\prod V_n$. Clearly, $d_n = \frac{d'}{d'+1}$ satisfies this assumption. This also implies that a Cauchy sequence for $d'$ is also a Cauchy sequence for $\frac{d'}{d'+1}$ (and more generally that both metrics are equivalent), so that $V$ is also a complete space with respect to $\frac{d'}{d'+1}$.

We're now ready to turn to the infinite product space $V^{\mathbb{N}}$ on which we define $d = \sum_n 2^{-n} \frac{d'}{d'+1}$ where $d'$ is the metric induced from the norm on $V$. We first claim that if $\{x^j\} \subset V^{\mathbb{N}}$ is a Cauchy sequence, then $\{x_m^j\} \subset V$ is a Cauchy sequence for every $m$. Indeed, we have for any $\varepsilon > 0$ there exists $C \in \mathbb{N}$ such that for all $i, j > C$ it holds that $\sum_n 2^{-n} \frac{d'(x_n^i, x_n^j)}{d'(x_n^i, x_n^j)+1} < \varepsilon$. It particularly implies that $\frac{d'(x_n^i, x_n^j)}{d'(x_n^i, x_n^j)+1} \leq 2^n \varepsilon$ for all such $i$ and $j$, and thus $\{x_n^i\}$ defines a Cauchy sequence for all $n$, and converges to an element $x_n \in V$. Therefore, for all $\varepsilon' > 0$ and all $N$ there exists $M$ such that for all $n < N$ and all $i > M$ one finds $\frac{d'(x_n^i, x_n)}{d'(x_n^i, x_n)+1} \leq \varepsilon'$. We put $\varepsilon' = \varepsilon/2N$ to bound the sum

$$\sum_{n<N} 2^{-n} \frac{d'(x_n^i, x_n)}{d'(x_n^i, x_n) + 1} < \frac{\varepsilon}{2}$$

for all $i > M = M(N, \varepsilon)$. On the other hand,

$$\sum_{n \geq N} 2^{-n} \frac{d'(x_n^i, x_n)}{d'(x_n^i, x_n) + 1} \leq \sum_{n \geq N} 2^{-n}$$

gets arbitrarily small as $N$ grows. Given $\varepsilon > 0$ we choose N such that this sum is bounded by $\frac{\varepsilon}{2}$. Then choose $M$ as above, to get

$$d((x_n^i), (x_n)) = \sum_{n<N} 2^{-n} \frac{d'(x_n^i, x_n)}{d'(x_n^i, x_n) + 1} + \sum_{n \geq N} 2^{-n} \frac{d'(x_n^i, x_n)}{d'(x_n^i, x_n) + 1} \leq \varepsilon$$

for all $i > M$.

**3.** a) Assume first that $f \in C_0$. The function $|f|$ is continuous so that the pre-image of $[\epsilon, \infty)$ is closed, and is inside $\Omega$ (since $f$ vanishes on $\partial\Omega$). As $\Omega$ is bounded, so must be the pre-image and therefore it is also compact.

Assume now that the set $|f|^{-1}(\{x \geq \epsilon\})$ is compact and let $y \in \partial\Omega$. We need to show that $f$ can be extended to $y$ with value zero. By compactness, $|f|^{-1}(\{x \geq \epsilon\}) \subset \Omega^\circ$ has (strict) positive distance to $\partial\Omega$, say $\delta$. The image of a small neighborhood around $y$, say $B_{\frac{\delta}{2}}(y) \cap \Omega$, under $f$ is therefore bounded by $\epsilon$. In other words, for any $\epsilon$ we found a neighborhood $A$ of $y$ in which $|f(x)| < \epsilon$. This implies that $f$ is continuously extendable by setting $f(y) = 0$.

b) Note that for the claim $\overline{C_c(\overline{\Omega}, \mathbb{R})} \subset C_0(\overline{\Omega}, \mathbb{R})$ it is enough to show that $C_0(\overline{\Omega}, \mathbb{R})$ is closed in $C(\overline{\Omega}, \mathbb{R})$. We show that the complement in $C_0$ is open. Suppose $f$ continuous on $\Omega$ with continuous extension that archieves a value strictly bigger then 0 on the boundary, say $f(y) > 0$. Hence any function $g$ such that $\|g - f\|_\infty < \frac{f(y)}{2}$ will also not vanish in $y$.

For density suppose $f \in C_0(\overline{\Omega}, \mathbb{R})$. Assume one has a sequence of continuous, positive functions $g_n : \Omega \to [0, 1]$ which satisfy $g_n|\Omega_n = 1$ where $\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > n^{-1}\}$ and vanish on $\Omega \setminus \Omega_{n+1}^c$. By continuity of $f$ and pre-compactness of $\Omega$ there exists $N$ such that for all $n > N$ one has $|f|_{\Omega_n^c}| < \varepsilon$. This implies that the function $g_n f \in C_c$ converges to $f$ as $n \to \infty$. To construct such a sequence $g_n$ one might use the indicator function over $\Omega_n$ and convolves with a continuous bump function with small support.

c) Since we've shown that $C_0$ is closed, one only needs to now that the space of bounded continuous functions is complete. This is example 2.12(2) and 2.12(3) in Chapter 2.2.1 of the Script.

**4.** Chapter 2.4.1.