Solution 3

1. Suppose $r < s$. We apply Stone-Weierstrass (with respect to the supremums norm on $C([0,s])$) to see that we can approximate a bumb function that vanishes on $[0,r]$ and archieves value 1 in (r,s) by polynomials: Thus for any $\epsilon > 0$ there is a $p \in \mathbb{R}[X]$ such that $\|p\|_r < \epsilon$ and $\|p\|_s = 1$. Clearly, these norms can't be equivalent. Stone-Weierstrass not necessary in order to give another easy argument: The polynomial $(\frac{x}{s})^n$ will be very small on $[0,r]$ for large n, while its norm in $C([0,s])$ is still 1.

2. First not that l^∞ is a complete space (script 2.19(7b)), so that any Cauchy-Sequence in c_c has a limit in l^∞. We construct a Cauchy-Sequence x^n who's limit is not in c_c. In fact the sequence $x^n = P_n(x)$ will do it. Here x is defined by $x_j = \frac{1}{j}$ (starting at $j = 1$) and P_n denotes the projection to the first n components, $P_n(x)_j = x_j$ for $j \leq n$ and $P_n(x)_j = 0$ for $j > n$. Then for any $\varepsilon > \frac{1}{m_0}$ we have for all $n \geq m > m_0$ that $\|x^n - x^m\|_\infty = \sup_{i \geq n} \frac{1}{i} = \frac{1}{n} > \frac{1}{m_0} > \varepsilon$ so that x^n is indeed Cauchy. But $x^n = P_n(x) \to x$ which is not in l^∞. But we not that x is a null-sequence. This is no coincidence: Let c_0 denote the space of null-sequences. Then $c_c \subset c_0 \subset l^\infty$. As we have done in the last exercise of the last sheet, to argue that c_0 is the completion of c_c we have to show that c_c is dense in c_0 and c_0 is closed in l^∞. The first fact follows again by using the projection operator P_n as we just did. For closedness assume that x^n is a sequence of null-sequences converging to an element x in l^∞. Let us restrict to a subsequence, again denoted by x^n for which $\|x^n - x\| < \frac{1}{n}$. Let $\varepsilon > \frac{2}{n}$ be arbitrary. Then we can calculate that

$$|x_j| \leq |x_j^n| + |x_j^n - x_j| < |x_j^n| + \|x^n - x\| < \frac{1}{n} + |x_j^n|.$$

As x^n is assumed to be a null sequence, for all j large enough $|x_j^n| \leq \frac{1}{n}$ and thus $|x_j| < \frac{2}{n} < \varepsilon$ showing that x is a null sequence.

3. The set A is closed since for any sequence x^n with $x_{2j}^n = 0$ we also have $\lim_n x_{2j}^n = 0$. Similarly, if $y^n \in B$ converges to an element y then $y_{2j-1} = \lim y_{2j-1}^n = \lim j y_{2j}^n = j \lim y_{2j}^n = j y_{2j}$. Note that the "pointwise convergence" we used follows from $x^n \to x \Leftrightarrow \sum |x_j^n - x_j| \to 0 \Rightarrow |x_j^n - x_j| \to 0$.

To show that $A + B$ is not closed is not so trivial. Note that the the previous statements for A and B basically from the linearity of the defining properties (and "closedness" of the equality sign) of A respectively B. We want to show that the description of $A + B$ is different in nature.

Note that an element y of B also defines a sequence which is not only l^1-integrable but also integrable with the weight (k) in the sense that if we define z by $z_{2k} = y_{2k}$ and 0 else then $\sum k|z_k| < \infty$. The set $A + B$ includes these sequences as we might set $x = y - z$ which is an element in A, so that indeed $z \in A + B$. In fact if for an element $z \in A + B$ we have that $z_{2k+1} = 0$ then we automatically know that it must be convergent with respect to the above weight.

The basic sum which comes to mind where convergence is easily read of is the geometric series $\sum_k k^{-s}$. This sum converges if and only if $s > 1$ which is a "open" condition. So define $z_{2k}^n = \frac{1}{k^{2+n^{-1}}}$ and $z_{2k} = k^{-2}$. Then by the previous observation $z^n \in A + B$. Applying the integral test for series to $z^n - z$ shows that $z^n \to z$ in l^1, but z is not integrable with respect to the weight (k) and thus $z \notin A + B$.

Please turn over!

4. 1. This is a subcase of part (3), viewing sequences as functions from $\mathbb{N}$ with the discrete topology. Notice though that the notion of (equi-)continuity on a discrete metric space is vacuous. Thus a characterization is: A subset is compact if and only if it is closed, bounded, and if for every e > 0, there exists N so that for any sequence x in the subset and for all $n \geq N$, $|x_n| \leq$ e.

2. We claim that a subset $A \subseteq l^p$ is compact if and only if A is closed, bounded, and for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $\|x|_N\|_p := \sum_{i=N}^{\infty} \|x_i\|_p < \varepsilon$.

   Let $\varepsilon > 0$ and let apply compactness of A to find a finite F subset of A such that $A \subset \cup_{x \in F} B_\varepsilon(x)$. As $\|x\|_p = (\sum |x_i|^p)^{1/p} < \infty$ there exists N_x such that $\|x|_{N_x}\|_p < \varepsilon$. Let $N = \max_{x \in F} N_x$. Then for any $y \in A$ we find $\|y|_N\|_p \leq \|(y-x)|_N\|_p + \|x|_N\|_p < \|y-x\|_p + \|x|_N\|_p < 2\varepsilon$. Compactness clearly implies boundedness and closedness.

   For the other direction, we want to find, given an arbitrary sequence $x^n \in l^p$, a Cauchy subsequence. Let $\varepsilon > 0$ and let N be such that $\|x|_N\|_p := \sum_{i=N}^{\infty} \|x_i\|_p < \varepsilon$ for all $x \in A$ which by assumption exists. But $(x_i^m)_{0 \leq i \leq N-1}$ is a sequence in $\mathbb{R}^N$, by boundedness of A in the l^p norm one also can also bound each component, such that $(x_i^m)_{0 \leq i \leq N-1}$ lies in a bounded set in $\mathbb{R}^N$. The same fact also implies that if A is closed in l^p then restricted to $\mathbb{R}^N$ it must be closed as well. By the Heine-Borel theorem for euclidean space there exists Cauchy subsequence of x^{m_k} with respect to $l^p(\mathbb{R}^N)$. Thus there exists k_ε such that x^{m_k} and x^{m_l} only differ by ε for all l and k greater than k_ε. As the tails of x^{m_k} only differ by ε these elements only differ by 2ε (using the triangle inequality for l^p). For $\varepsilon_m = m^{-1}$ we can repeat this iteratively to construct a Cauchy sequence (as we do in the next part of the exercise).

3. A subset $A \subseteq C_0(X)$, where X is a separable locally compact metric space, is compact if and only if A is closed, bounded, equicontinuous when restricted to any compact set, and uniformly vanishing at infinity (that is, for every e > 0 there exists a compact set $K \subset X$ such that for all $x \notin K$ and for all $f \in A$, $|f(x)| <$ e).

   The proof of this fact is a slight variation of the proof of the Arzela-Ascoli Theorem.

   Suppose that $A \subset C_0(X)$ is compact, from which immediately follows that it is closed and bounded. For any compact set K of X we can consider the set A also as subset of $C(K)$ for which we can apply the Arzela-Ascoli theorem for compact metric spaces to see that A when restricted to any compact set is also equicontinous. To show that the elements of A vanish uniformly at infinity, let e > 0. Then $\bigcup_{f \in A} B_{\frac{e}{2}}(f)$ is an open cover of A and by compactness there exists a finite subcover, say $A \subseteq \bigcup_{i=1}^{n} B_{\frac{e}{2}}(f_i)$. By the description of C_0 given in the last serie, for each i there exists a compact set $K_i \subseteq X$ such that

$$\sup_{x \in X \setminus K_i} |f_i(x)| < \frac{e}{2}.$$

   Define $K = \bigcup K_i$, also a compact set. Then for $x \in X \setminus K$ and $f \in A$, we have for some i

$$|f(x)| \leq |f(x) - f_i(x)| + |f_i(x)| < e.$$

   As for the other direction, let $f_n \in A$ be a arbitrary sequence. We will show that f_n has a Cauchy-subsequence (and by closedness of A and completeness of $C(X)$ a convergent subsequence). Let $\varepsilon > 0$ be arbitrary and by the uniform vanishing tail assumption there exists a compact K for which $\sup_{x \in K^c} |f(x)| < \varepsilon$ for all $f \in A$. Apply the Arzela-Ascoli theorem to A seen as functions on $C(K)$ to see that there is a subseqence f_{n_k} that is Cauchy with respect to $\|\cdot\|_{(C(K)}$. Thus there exists k_ε such that for all $l, k \geq k_0$, the functions f_{n_l} and f_{n_k} are ε close on K, and thus ε-close on all of X. We apply algorithm for each $\varepsilon_m = m^{-1}$ iteratively (call the subsequence we get in each iteration f_n^m), and by defining $g_m = f_{k_{\varepsilon_m}}^m$ we get a Cauchy sequence in X.