Solution 12

1. Since U and V are assumed to be closed in B, they are also Banach spaces. By assumption $\Psi : (u, v) \mapsto u + v$ is bijective, and also continuous by the triangle inequality. Thus Proposition 5.18 about the existence of a bounded inverse applies, that is, Ψ is an Banach space isomorphism.

2. **a)** implies b) by the open mapping theorem.

b) implies c) which we see, when we imply openess at the point $0 \in Y$: There exists ε such that $B_\varepsilon^Y \subset TB_1^X$. Thus if $y \in Y$ is arbitrary, $\frac{\varepsilon y}{\|y\|} \in TB_1^X$, that is, there exists x such that $Tx = y$ and $\|x\| \leq \frac{1}{\varepsilon}\|y\|$.

c) implies a) since a) is the qualitative statement of c), and surely also implies d).

d) implies c): If $y \in Y$ arbitrary and Y' be the dense quantitatively solvable set. Let $y_n \in Y'$ be such that $\sum y_n = y$. Then $\sum x_n = T \sum y_n = Ty$. On the otherhand, $\sum x_n \leq C\| \sum y_n\|$ and the latter sum is absolute convergent, thus the former sum is. By completeness of X, $\sum x_n$ converges to an element in X.

3. **a)** A standard fact from analysis but we add the proof (Rudin, Mathematical analysis). Since any convergent sequence is Cauchy, $\|f_n - f_m\| \leq \varepsilon$ for $n, m > N$, and similiarly $\|f_n' - f_m'\| \leq \varepsilon$. Define $D_h(F) = \frac{F(x+h)-F(x)}{h}$, then $\lim_{h\to 0} D_h f_n(x) = f_n'(x)$ for each n and x. Note that f_n is Lipschitz with Lipschitz constant $2\|f'\|$. Similiarly, $f_n - f_m$ is Lipschitz with constant $2\|f_n' - f_m'\| \leq 2\varepsilon$. Thus $\|D_h f_n - D_h f_m\| \leq 2\varepsilon$ for all h and $n, m > N$ which shows that $D_h f_n$ is Cauchy and thus converges uniformly, and this limit is $D_h f$. As $h \to 0$, $D_h f_n \to f_n'$ and $D_h f \to f'$. By the usual 3ε argument we can interchange the limit of h and n so that $f' = \lim_h D_h f = \lim_h \lim_n D_h f_n = \lim_n \lim_h D_h f_n = \lim_n f_n' = g$.

b) The function $f_n(x) = (x^2 + \frac{1}{n})^{\frac{1}{2}}$ converges uniformly to $f(x) = |x|$ (and thus in L^2) and one checks that f_n' converges to a stepfunction L^2. But f is not continuously differentiable, which implies that the derivative is not a closed operator.

c) H^1 is a closure of the graph of the derivative of C^1 in L^2.