Solution 13

1. **a)** Pick a normalized basis $\{v_i\}$ of the finite dimensional subspace V and let $\{v_i^*\}$ be the corresponding dual basis, so that we have $v_i^*(v_j) = \delta_{ij}$. These are continuous functionals of norm one and we may extend them to functionals $\tilde{v}_i^*$ on all of X by Hahn-Bahnach. Then $Px = \sum_{i=1}^{\dim V} \tilde{v}_i^*(x)v_i$ defines a projection and we may apply Serie 7 exercise 4.

b) Let $\lambda_1, \lambda_2 \in X^*$ be two extension of λ. By normalizing we may assume that $\|\lambda\| = 1$ such that $\|\lambda_1\| = \|\lambda_2\| = 1$. Note that by linearity, $\frac{\lambda_1+\lambda_2}{2}$ also defines an extension of λ. But strict convexity gives

$$\left\|\frac{\lambda_1 + \lambda_2}{2}\right\| < \frac{\|\lambda_1\|}{2} + \frac{\|\lambda_2\|}{2} = 1$$

if we would have $\lambda_1 \neq \lambda_2$. But this contradicts that $\lambda = 1$ hence we must have $\lambda_1 = \lambda_2$.

c) One example would be the evaluation map at the point $x \in [0, 1]$ on $C([0, 1], \|\cdot\|_\infty)$ restricted to the space of constant functions. Note that they all have norm one. This restriction is independent of x hence any evaluation would extend it.

d) Let $\{x_n^*\}$ be countable dense subset of X^*. For each $x^* \in X^*$ we may choose a sequence of unit vectors $x^{(k)}$ such that $x^*(x^{(k)}) \to \|x^*\|$ by definition of the (operator-)norm on X^*. In particular, we find for any n some $x_n \in X$ such that $x_n^*(x_n) \geq \frac{\|x^*\|}{2}$. The set of all finite linear combinations with rational coefficients $\operatorname{span}_{\mathbb{Q}}\{x_n\}$ is countable. We need to prove that it is in fact dense. Denote Y the closure of this set (this surely is also the closure of the $\mathbb{R}$-span). Assume by contradiction that $Y \neq X$ then by the Hahn-Banach-Theorem there exists a nontrivial linear functional $x^* \in X^*$ such that x^* vanishes on Y. By density of the x_n^*'s we may choose n_0 such that

$$\|x^* - x_{n_0}^*\| < \varepsilon.$$

This would also imply that $x_{n_0}^*(x_{n_0}) = |x^*(x_{n_0}) - x_{n_0}^*(x_{n_0})| < \varepsilon$ since x_{n_0} is inside the unit ball. This contradicts the definition of x_{n_0}.

2. **a)** Define $\Phi : \ell^q \to (\ell^p)^*$ by

$$\Phi(f)(g) = \varphi_f(g) = \sum f(n)g(n)$$

for any $f \in \ell^q$ and $g \in \ell^p$. We claim that Φ is an isometric isomorphism. Let $f \in \ell^q$. By Hölder's inequality

$$|\varphi_f(g)| \leq |\sum f(n)g(n)| \leq \|f\|_q\|g\|_p.$$

Thus $\|\varphi_f\| \leq \|f\|_q$ showing $\varphi_f \in (\ell^p)^*$. To show equality (and that Φ is an isometry), define the finite sequence $g_m = f\chi_{[1,m]}$, and consider

$$\tilde{g}_m = \frac{g_m^{q-1}\operatorname{sgn}(g_m)}{\|g_m\|_q^{q-1}}$$

where we are using pointwise multiplication of functions. Then

$$\|\tilde{g}_m\|_p^p = \frac{\sum_{i=1}^m |f(i)|^{(q-1)p}}{\left(\sum_{i=1}^m |f(i)|^q\right)^{\frac{q-1}{q}p}} = 1$$

because $(q-1)p = q$. Moreover

$$|\varphi_f(\tilde{g}_m)| = \left| \frac{1}{\|g_m\|_q^{q-1}} \sum_{i=1}^m |f(i)|^q \right| = \left| \left(\sum_{i=1}^m |f(i)|^q \right)^{1-\frac{1}{p}} \right| = \|f\chi_{[1,m]}\|_q \nearrow \|f\|_q.$$

Thus, Φ is an isometry as claimed.

It remains to show that Φ is surjective. Let $\varphi \in (\ell^p)^*$, and define $f(n) = \varphi(\chi_{\{n\}})$. We wish that $f \in \ell^q$. Define g_m and $\tilde{g}_m$ for this f as before. Then

$$\|f\chi_{[1,m]}\|_q = |\varphi(\tilde{g}_m)| \leq \|\varphi\| \|f\|_q.$$

Since this estimate is independent of m, $\|f\|_q < \infty$, giving the desired result.

b) First assume that (f_n) converges weakly to a point f in ℓ^p. By replacing (f_n) with $(f_n - f)$, we may assume that f_n converges weakly to 0. Then consider (f_n) as a collection of functionals on ℓ^q where q is the conjugate to p. Fix $g \in \ell^q$. Then duality implies

$$|\varphi_{f_n}(g)| = |\varphi_g(f_n)| \to 0 \text{ as } n \to \infty.$$

In particular, $\sup_n |\varphi_{f_n}(g)| < \infty$. Thus, we may apply Banach-Steinhaus to conclude that $\sup_n \|f_n\|_p = \sup_n \|\varphi_{f_n}\| < \infty$. Thus, the sequence is bounded in norm. To see pointwise convergence to 0, apply the continuous linear functional associated to the sequence $\chi_{\{i\}}$ for each i.

Now assume that $(f_n) \subset \ell^p$ such that there exists B with $\|f_n\|_p \leq B$ and there exists $f \in \ell^p$ with $f_n(i) \to f(i)$ for each i. Replacing f_n by $f_n - f$, we may assume pointwise convergence to 0. Let φ be a continuous linear functional on ℓ^p, that is $\varphi = \varphi_g$ for some $g \in \ell^q$ where q is conjugate to p. Then for $m \in \mathbb{N}$,

$$|\varphi_g(f_n) - \varphi_{g\chi_{[1,m]}}(f_n)| = |\varphi_{g\chi_{(m,\infty)}}(f_n)| \leq \|g\chi_{(m,\infty)}\|_q \|f_n\|_p \leq \|g\chi_{(m,\infty)}\|_q B \to 0$$

as $m \to \infty$. Thus independently of n there exists M such that $|\varphi_g(f_n) - \varphi_{g\chi_{[1,M]}}(f_n)|$ is small. By pointwise convergence to 0, there exists N so that for all $n > N$, $|\varphi_{g\chi_{[1,M]}}(f_n)|$ is small. Thus, for large n, $|\varphi_g(f_n)|$ is small, giving weak convergence to 0.

c) Consider the sequence $n^{-\frac{1}{p}}\chi_{[1,n]}$. Then this sequence converges to zero pointwise and $\|n^{-\frac{1}{p}}\chi_{[1,n]}\|_p = 1$ for all n. Then by part (a), this sequence converges weakly to 0. However since the norm is constantly 1, the sequence does not converge in norm to 0.