D-MATH Geometric Approximation Theory HS 2014
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Exercise Sheet 3

1. Exponential, Logarithm and Injectivity Radius on the Sphere

a) Prove that the exponential and the logarithm function on S^m for $p, q \in S^m$ and $r \in T_pS^m$ with $p \neq q$ and $r \neq 0$ are

$$\begin{aligned}\exp_p(r) &= \cos(\|r\|_2)p + \frac{\sin(\|r\|_2)}{\|r\|_2} r \text{ and} \\ \log_p(q) &= \frac{\arccos(\langle p,q\rangle)}{\sqrt{1-\langle p,q\rangle^2}} \left(q - \langle p,q\rangle p\right).\end{aligned}$$

b) What is the injectivity radius on S^m?

2. Derivative of Squared Distance Function

Let $\mathcal{M}$ be a Riemannian submanifold of $\mathbb{R}^n$ and $d\colon \mathcal{M}\times\mathcal{M} \to \mathbb{R}_{\geq 0}$ its distance function. The square of the the distance function will play an important role for defining approximations of manifold-valued functions. In this exercise we will compute the derivative of the squared distance function. Let $a \in \mathcal{M}$, r the injectivity radius of a, $y \in \mathcal{M}$ with $d(a,y) < r$, $\gamma\colon [0,1] \to \mathcal{M}$ a smooth curve with $\gamma(0) = y$ and $d(\gamma(t), a) < r$ for all $t \in [0,1]$. Furthermore let $c\colon [0,1]^2 \to M$ be defined by

$$c(s,t) := \exp_a(s \log_a(\gamma(t))).$$

Denote

$$c' := \frac{d}{ds}c(s,t) \quad \text{and} \quad \dot{c} := \frac{d}{dt}c(s,t).$$

a) Show that

$$\frac{d}{dt}\int_0^1 \langle c'(s,t), c'(s,t)\rangle ds = 2\langle \dot{c}(1,t), c'(1,t)\rangle$$

b) Conclude that

$$\frac{d}{dt} d(a,\gamma(t))^2 = -2\langle \dot{\gamma}(t), \log_{\gamma(t)}(a)\rangle$$

and

$$\operatorname{grad} d(a,y)^2 = -2\log_y(a)$$

where for a function $f\colon \mathcal{M} \to \mathbb{R}$ and $x \in \mathcal{M}$ we define $\operatorname{grad} f$ as the unique element such that

$$\langle \operatorname{grad} f, \xi \rangle = Df[\xi]$$

for all $\xi \in T_x\mathcal{M}$.

c) Determine where the function $y \mapsto d(a, y)$ is differentiable and compute the gradient.

3. Riemannian Quotient Manifold

Let $\overline{\mathcal{M}}$ be a Riemannian manifold with Riemannian metric $\overline{g}$, $\sim$ a regular equivalence relation and $\mathcal{M} = \overline{\mathcal{M}}/\sim$. We choose the horizontal space $H_{\overline{x}}$ at $\overline{x} \in \overline{\mathcal{M}}$ as the orthogonal complement of the vertical space $\mathcal{V}_{\overline{x}}$. For $x = \pi(\overline{x})$, $\xi_x \in T_x\mathcal{M}$ let $\overline{\xi}_{\overline{x}}$ be the horizontal lift at $\overline{x}$. If, for every $x \in M$ and every $\xi_x, \zeta_x \in T_x\mathcal{M}$ the expression $\overline{g}_{\overline{x}}(\overline{\xi}_{\overline{x}}, \overline{\zeta}_{\overline{x}})$ does not depend on $\overline{x} \in \pi^{-1}(x)$, then

$$g_x(\xi_x, \zeta_x) := \overline{g}_{\overline{x}}(\overline{\xi}_{\overline{x}}, \overline{\zeta}_{\overline{x}})$$

defines a Riemannian metric on $\mathcal{M}$. Consider now the Riemannian metric

$$\overline{g}_Y(Z_1, Z_2) = trace((Y^TY)^{-1}Z_1^TZ_2)$$

on $\mathbb{R}_*^{n\times p}$. Show that $\overline{g}$ induces a Riemannian metric on $\mathbb{R}_*^{n\times p}/GL_p$.

To be submitted on October 20.