ETH Zürich HS 2014
D-MATH
Prof. Dr. Johannes Muhle-Karbe

Coordinator
Ren Liu

# Interest Rate Theory
## Solution Sheet 6

**1.** **a)** In the Vasiček short-rate model using the affine term-structure and

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T}\log(P(t,T)) \\ &\stackrel{ATS}{=} \frac{\partial}{\partial T}A(t,T) + \frac{\partial}{\partial T}B(t,T)r(t) \end{aligned}$$

we can obtain an explicit solution to the forward-rate process

$$f(t,T) = -(1-e^{\beta(T-t)})\frac{b}{\beta} - \frac{\sigma^2}{2\beta^2}(1-e^{\beta(T-t)})^2 + e^{\beta(T-t)}r(t)$$

If $\beta < 0$, then this converges to

$$\lim_{T\to\infty} f(t,T) = -(\frac{b}{\beta} + \frac{\sigma^2}{2\beta^2}),$$

and from L'Hôpital's rule also

$$R_\infty(t) = \lim_{T\to\infty}\frac{1}{T-t}\int_t^T f(t,s)ds = \lim_{T\to\infty} f(t,T).$$

**b)** In the CIR model we similarly have by the affine term structure that

$$f(t,T) = \frac{\partial}{\partial T}A(t,T) + \frac{\partial}{\partial T}B(t,T)r(t),$$

where setting $\gamma = \sqrt{\beta^2 + 2\sigma^2}$,

$$\begin{aligned} A(t,T) &= -\frac{2b}{\sigma^2}\log\left(\frac{2\gamma e^{(\gamma-\beta)(T-t)/2}}{(\gamma-\beta)(e^{\gamma(T-t)}-1)+2\gamma}\right), \\ \text{and} \quad B(t,T) &= \frac{2(e^{\gamma(T-t)}-1)}{(\gamma-\beta)(e^{\gamma(T-t)}-1)+2\gamma} \end{aligned}$$

$$
\begin{aligned}
X(T) &= e^{\gamma(T-t)} - 1 \\
Y(X(T)) &= (\gamma-\beta)(e^{\gamma(T-t)} - 1) + 2\gamma \\
Z(T) &= \gamma e^{(\gamma-\beta)(T-t)/2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_T X(T) = \gamma e^{\gamma(T-t)} \\
&\partial_T (Y(X(T))) = (\gamma-\beta)\partial_T X(T) = (\gamma-\beta)\gamma e^{\gamma(T-t)} \\
&\partial_T Z(T) = Z(T)\frac{\gamma-\beta}{2}
\end{aligned}
$$

thus

$$
\begin{aligned}
\frac{\partial}{\partial T}A(t,T) &= \frac{-2b}{\sigma^2}\log\left(\frac{2Z(T)}{Y(X(T))}\right) \\
&= \frac{-2b}{\sigma^2}\left(\frac{Y(X(T))}{2Z(T)}\right)\left(\frac{2\partial_T Z(T)}{Y(X(T))} - \frac{2Z(T)(\gamma-\beta)\partial_T X(T)}{(Y(X(T)))^2}\right) \\
&= \frac{-2b}{\sigma^2}\left(\frac{1}{2Z(T)}\right)\left(2\frac{\gamma-\beta}{2}Z(T) - \frac{2Z(T)(\gamma-\beta)\partial_T X(T)}{Y(X(T))}\right) \\
&= \frac{-b}{\sigma^2}\left((\gamma-\beta) - \frac{2(\gamma-\beta)\partial_T X(T)}{Y(X(T))}\right) \\
&= \frac{-b}{\sigma^2}\left((\gamma-\beta) - \frac{2(\gamma-\beta)\gamma e^{\gamma(T-t)}}{(\gamma-\beta)(e^{\gamma(T-t)} - 1) + 2\gamma}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial T}B(t,T) &= \frac{\partial}{\partial T}\left(\frac{2X(T)}{Y(X(T))}\right) \\
&= \frac{2\partial_T X(T)}{Y(X(T))} - \frac{2X(T)(\gamma-\beta)\partial_T X(T)}{(Y(X(T)))^2} \\
&= \frac{2\gamma e^{\gamma(T-t)}}{(\gamma-\beta)(e^{\gamma(T-t)} - 1) + 2\gamma} - \frac{2(e^{\gamma(T-t)} - 1)(\gamma-\beta)\gamma e^{\gamma(T-t)}}{((\gamma-\beta)(e^{\gamma(T-t)} - 1) + 2\gamma)^2}.
\end{aligned}
$$

For $\lim_{T\to\infty}$ the above terms converge to

$$
\begin{aligned}
\lim_{T\to\infty}\frac{\partial}{\partial T}A(t,T) &= \lim_{T\to\infty}\left(\frac{-b}{\sigma^2}\left((\gamma-\beta) - \frac{2(\gamma-\beta)\gamma e^{\gamma(T-t)}}{(\gamma-\beta)(e^{\gamma(T-t)} - 1) + 2\gamma}\right)\right) \\
&= \frac{-b}{\sigma^2}(\gamma-\beta) - \lim_{T\to\infty}\left(\frac{-b}{\sigma^2}\frac{2(\gamma-\beta)\gamma e^{\gamma(T-t)}}{(\gamma-\beta)(e^{\gamma(T-t)} - 1) + 2\gamma}\right) \\
&= \frac{-b}{\sigma^2}(\gamma-\beta) + \frac{2b\gamma}{\sigma^2} \\
&= \frac{b}{\sigma^2}(\gamma+\beta),
\end{aligned}
$$

**Siehe nächstes Blatt!**

and applying L'Hôpital's rule we get

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{\partial}{\partial T} B(t,T) \\
&= \lim_{T\to\infty} \left( \frac{2\gamma e^{\gamma(T-t)}}{(\gamma-\beta)(e^{\gamma(T-t)}-1)+2\gamma} \right) - \lim_{T\to\infty} \left( \frac{2X(T)(\gamma-\beta)\partial_T X(T)}{(Y(X(T)))^2} \right) \\
&= \frac{2\gamma}{(\gamma-\beta)} - \lim_{T\to\infty} \left( \frac{2(\gamma-\beta)\Big((\partial_T X(T))^2 + X(T)\gamma\partial_T X(T)\Big)}{2(Y(X(T)))(\gamma-\beta)\partial_T X(T)} \right) \\
&= \frac{2\gamma}{(\gamma-\beta)} - \lim_{T\to\infty} \left( \frac{\partial_T X(T) + X(T)\gamma}{Y(X(T))} \right) \\
&= \frac{2\gamma}{(\gamma-\beta)} - \lim_{T\to\infty} \left( \frac{\gamma e^{\gamma(T-t)} + (e^{\gamma(T-t)}-1)\gamma}{(\gamma-\beta)(e^{\gamma(T-t)}-1)+2\gamma} \right) \\
&= \frac{2\gamma}{(\gamma-\beta)} - \frac{2\gamma}{(\gamma-\beta)} = 0
\end{aligned}
$$

Therefore,

$$
\lim_{T\to\infty} f(t,T) = \frac{b}{\sigma^2}(\gamma+\beta),
$$

and as above, by L'Hôpital's rule also

$$
R_\infty(t) = \lim_{T\to\infty} \frac{1}{T-t} \int_t^T f(t,s)ds = \lim_{T\to\infty} f(t,T) = \frac{b}{\sigma^2}(\gamma+\beta).
$$

**c)** By assumption, the volatility process of the one-dimensional HJM-model is given by $\sigma(t,T) = (1+T-t)^{-1/2}$ for all $0 \le t \le T$. Then

$$
\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t \left( \sigma(s,T) \int_s^T \sigma(s,u)du \right) ds + \int_0^t \sigma(s,T)dW^*(s) \\
&= f(0,T) + \int_0^t \left( (1+T-s)^{-1/2} \int_s^T (1+u-s)^{-1/2} du \right) ds \\
&\quad + \int_0^t (1+T-s)^{-1/2} dW^*(s).
\end{aligned}
$$

In the limit the last term vanishes in probability

$$
\lim_{T\to\infty} \int_0^t (1+T-s)^{-1/2} dW^*(s) = 0.
$$

Also,

$$\begin{aligned}
&\int_0^t \left( (1+T-s)^{-1/2} \int_s^T (1+u-s)^{-1/2} du \right) ds \\
&= \int_0^t \left( (1+T-s)^{-1/2} \left( 2(1+T-s)^{1/2} - 2 \right) \right) ds \\
&= 2 \int_0^t 1 - (1+T-s)^{-1/2} ds \\
&= 2t - \int_0^t (1+T-s)^{-1/2} ds.
\end{aligned}$$

Thus the long rate for this model is

$$R_\infty(t) = \lim_{T\to\infty} f(t,T) = \lim_{T\to\infty} f(0,T) + 2t,$$

which is in fact strictly increasing in $t$. Note that the long rate exists, i.e. $\lim_{T\to\infty} f(t,T)$ converges if and only if $\lim_{T\to\infty} f(0,T)$ is finite.

**2.** (Bond Option Pricing in the Ho-Lee Model)

**a)** The arbitrage-free price at $t = 0$ of a European call option on an $S$-bond with expiry date $T < S$ and strike price $K$ is

$$\mathbb{E}_\mathbb{Q} \left[ e^{-\int_0^T r(s)ds} (P(T,S) - K)^+ \right], \tag{1}$$

where $\mathbb{Q}$ denotes the risk-neutral spot measure[1]. The above formula can be rewritten as

$$\begin{aligned}
&\mathbb{E}_\mathbb{Q} \left[ e^{-\int_0^T r(s)ds} (P(T,S) - K)^+ \right] \\
&= \mathbb{E}_\mathbb{Q} \left[ B(T)^{-1} (P(T,S) - K)^+ \right] \\
&= \mathbb{E}_\mathbb{Q} \left[ B(T)^{-1} P(T,S) \mathbb{I}_{P(T,S)>K} \right] - K \mathbb{E}_\mathbb{Q} \left[ B(T)^{-1} \mathbb{I}_{P(T,S)>K} \right].
\end{aligned}$$

Bayes' rule, with

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{1}{P(0,T)B(T)} \quad \text{and} \quad \mathbb{E}_\mathbb{Q} \left[ \frac{1}{P(0,T)B(T)} \right] = 1$$

yields

$$\begin{aligned}
K \mathbb{E}_\mathbb{Q} \left[ B(T)^{-1} \mathbb{I}_{P(T,S)>K} \right] &= K P(0,T) \mathbb{E}_\mathbb{Q} \left[ \frac{1}{P(0,T)B(T)} \mathbb{I}_{P(T,S)>K} \right] \\
&= K P(0,T) \mathbb{Q}^T \left[ P(T,S) > K \right],
\end{aligned}$$

[1] Cf. p.106. Filipović, T.S.M.

**Siehe nächstes Blatt!**

and similarly for

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{1}{P(0,S)B(S)} \quad \frac{d\mathbb{Q}^S}{d\mathbb{Q}}\Big|_{\mathcal{F}_T} = \frac{P(T,S)}{P(0,S)B(T)}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[B(T)^{-1}P(T,S)\mathbb{I}_{P(T,S)>K}\right] \\
&= P(0,S)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{P(0,S)B(T)}P(T,S)\mathbb{I}_{P(T,S)>K}\right] \\
&= P(0,S)\mathbb{E}_{\mathbb{Q}^S}\left[\mathbb{I}_{P(T,S)>K}\right] \\
&= P(0,S)\mathbb{Q}^S\left[P(T,S)>K\right].
\end{aligned}$$

Furthermore,

$$\mathbb{Q}^T\left[P(T,S)>K\right] = \mathbb{Q}^T\left[\frac{P(T,S)}{P(T,T)}>K\right], \tag{2}$$

$$\text{and} \quad \mathbb{Q}^S\left[P(T,S)>K\right] = \mathbb{Q}^S\left[\frac{P(T,T)}{P(T,S)}<\frac{1}{K}\right], \tag{3}$$

and it remains to show that the random variables $\frac{P(T,S)}{P(T,T)}$ and $\frac{P(T,T)}{P(T,S)}$ both have a lognormal distribution and to determine the corresponding mean and variance. By Lemma 7.1, for any $t \leq T \wedge S$

$$\begin{aligned}
\frac{P(t,S)}{P(t,T)} &= \frac{P(0,S)}{P(0,T)}\mathcal{E}_t\left(\sigma_{S,T} \bullet W^T\right) \\
&= \tfrac{P(0,S)}{P(0,T)} e^{\int_0^t \sigma_{S,T}(s)dW^T(s) - \frac{1}{2}\int_0^t \sigma_{S,T}(s)^2 ds}.
\end{aligned} \tag{4}$$

We apply (4) with $t = T$ to obtain (2), and to obtain (3) interchange the role of $T$ and $S$. Now we note that $\sigma_{S,T}(s) = -\sigma_{T,S}(s)$ for all $s > 0$ and determine the function $\sigma_{S,T}(s)$ corresponding to the Ho-Lee Model. Recall from the lecture that for the Ho-Lee model it holds $\sigma(s,u) \equiv \sigma$, which yields (cf. Lemma 7.1) that

$$\sigma_{S,T}(s) = \int_S^T \sigma(s,u)du = (T-S)\sigma \quad \text{for all} \quad T < S.$$

**Bitte wenden!**

Then

$$\begin{aligned}
\mathbb{Q}^T\left[\frac{P(T,S)}{P(T,T)} > K\right] &= \mathbb{Q}^T\left[\tfrac{P(0,S)}{P(0,T)}e^{\int_0^T \sigma_{S,T}(s)dW^T(s) - \frac{1}{2}\int_0^T \sigma_{S,T}(s)^2 ds} > K\right] \\
&= \mathbb{Q}^T\left[\tfrac{P(0,S)}{P(0,T)}e^{(T-S)\sigma W^T(T) - \frac{1}{2}(T-S)^2 T\sigma^2} > K\right] \\
&= \mathbb{Q}^T\left[W^T(T) < \frac{\log\left(\frac{P(0,T)}{P(0,S)}K\right)}{(T-S)\sigma} + \frac{1}{2}(T-S)T\sigma\right] \\
&= \mathbb{Q}^T\left[\frac{W^T(T)}{\sqrt{T}} < \frac{\log\left(\frac{P(0,S)}{P(0,T)K}\right)}{\sqrt{T}(S-T)\sigma} - \frac{1}{2}(S-T)\sqrt{T}\sigma\right]
\end{aligned}$$

where $W^T(T) \sim \mathcal{N}(0, \sqrt{T})$, hence $\frac{W^T(T)}{\sqrt{T}} \sim \mathcal{N}(0,1)$ under $\mathbb{Q}^T$.
Similarly, since $\frac{W^S(T)}{\sqrt{T}} \sim \mathcal{N}(0,1)$ under $\mathbb{Q}^S$,

$$\begin{aligned}
\mathbb{Q}^S\left[\frac{P(T,T)}{P(T,S)} < \frac{1}{K}\right] &= \mathbb{Q}^S\left[\tfrac{P(0,T)}{P(0,S)}e^{\int_0^T \sigma_{T,S}(s)dW^S(s) - \frac{1}{2}\int_0^T \sigma_{T,S}(s)^2 ds} < \frac{1}{K}\right] \\
&= \mathbb{Q}^S\left[\tfrac{P(0,T)}{P(0,S)}e^{(S-T)\sigma W^T(T) - \frac{1}{2}(S-T)^2 T\sigma^2} < \frac{1}{K}\right] \\
&= \mathbb{Q}^S\left[\frac{W^S(T)}{\sqrt{T}} < \frac{\log\left(\frac{P(0,S)}{P(0,T)K}\right)}{\sqrt{T}(S-T)\sigma} + \frac{1}{2}(S-T)\sqrt{T}\sigma\right]
\end{aligned}$$

The price $c(0,T,S,K)$ of a European call option at time $t = 0$ with strike price $K$ and exercise date $T$ on an underlying $S$-bond, where $T < S$ is of the form

$$c(0,T,S,K) = P(0,S)\Phi(d_1) - P(0,T)K\Phi(d_2),$$

where $\Phi$ denotes the cumulative normal distribution, and $d_1$ and $d_2$ are the values

$$\begin{aligned}
d_{1,2} &= \frac{1}{\sigma_p}\log\left(\frac{P(0,S)}{P(0,T)K}\right) \pm \frac{1}{2}\sigma_p \\
\text{with} \quad \sigma_p &= \sigma(S-T)\sqrt{T}.
\end{aligned} \tag{5}$$

**b)** For a European put option on an $S$-bond with expiry date $T < S$ and strike price

**Siehe nächstes Blatt!**

$K$ we have the following $t = 0$ arbitrage-free price

$$\begin{aligned}
p(0,T,S,K) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}(K - P(T,S))^+\right] \\
&= K\mathbb{E}_{\mathbb{Q}}\left[B(T)^{-1}\mathbb{I}_{P(T,S)<K}\right] - \mathbb{E}_{\mathbb{Q}}\left[B(T)^{-1}P(T,S)\mathbb{I}_{P(T,S)<K}\right] \\
&= KP(0,T)\mathbb{Q}^T\left[P(T,S) < K\right] - P(0,S)\mathbb{Q}^S\left[P(T,S) < K\right] \\
&= KP(0,T)\mathbb{Q}^T\left[\frac{P(T,S)}{P(T,T)} < K\right] - P(0,S)\mathbb{Q}^S\left[\frac{1}{K} < \frac{P(T,T)}{P(T,S)}\right],
\end{aligned}$$

and by analogous arguments to the above we get

$$p(0,T,S,K) = KP(0,T)\Phi(-d_2) - P(0,S)\Phi(-d_1)$$

with $d_{1,2}$ as in (5).

**3.** See capvasicekatm.m and fittedsigma.m

**4.** We first recall the derivation of the caplet price. Let $t \in [0, T_{m-1}]$ be arbitrary. The time $t$-caplet price is given by

$$\mathrm{Cpl}(t; T_{m-1}, T_m) = P(t, T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}[\delta(L(T_{m-1}, T_{m-1}) - \varkappa)^+ \mid \mathcal{F}_t].$$

Noting that

$$\begin{aligned}
&L(T_{m-1}, T_{m-1}) = L(t, T_{m-1})\times \\
&\times \exp\bigg(-\frac{1}{2}\int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds + \int_t^{T_{m-1}} \lambda(s, T_{m-1})dW_s^{T_m}\bigg),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\log L(T_{m-1}, T_{m-1}) = \log L(t, T_{m-1}) \\
&-\frac{1}{2}\int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds + \int_t^{T_{m-1}} \lambda(s, T_{m-1})dW_s^{T_m}.
\end{aligned}$$

Hence, the $\mathbb{Q}^{T_m}$-distribution of $\log L(T_{m-1}, T_{m-1})$ conditional on $\mathcal{F}_t$ is Gaussian with mean

$$\mu = \log L(t, T_{m-1}) - \frac{1}{2}\int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds$$

and variance

$$\sigma^2 = \int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds.$$

**Bitte wenden!**

In particular

$$\begin{aligned}\mu + \frac{\sigma^2}{2} &= \log L(t, T_{m-1}),\\ \mu + \sigma^2 &= \log L(t, T_{m-1}) + \frac{1}{2}\int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds.\end{aligned}$$

Recall[2] that if $X \sim N(\mu, \sigma^2)$ is a normally distributed random variable and $c \in \mathbb{R}$ is a constant, then we have

$$\begin{aligned}\mathbb{P}(X \geq c) &= \Phi\Big(-\frac{c-\mu}{\sigma}\Big),\\ \mathbb{E}[e^X \mathbb{1}_{\{X \geq c\}}] &= e^{\mu + \frac{\sigma^2}{2}}\Phi\Big(-\frac{c-(\mu+\sigma^2)}{\sigma}\Big),\end{aligned}$$

where $\Phi$ denotes the standard Gaussian cumulative distribution function. Therefore,

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[2]Here is the proof: We have $\frac{X-\mu}{\sigma} \sim N(0,1)$, and hence

$$\mathbb{P}(X \geq c) = \mathbb{P}\Big(\frac{X-\mu}{\sigma} \geq \frac{c-\mu}{\sigma}\Big) = \mathbb{P}\Big(\frac{X-\mu}{\sigma} \leq -\frac{c-\mu}{\sigma}\Big) = \Phi\Big(-\frac{c-\mu}{\sigma}\Big). \tag{6}$$

The density of the normal distribution is given by

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

Therefore, we have

$$\mathbb{E}[e^X \mathbb{1}_{\{X \geq c\}}] = \int_c^\infty e^x f_{\mu,\sigma^2}(x)dx = \int_c^\infty \frac{1}{\sqrt{2\pi\sigma^2}}e^{x-\frac{(x-\mu)^2}{2\sigma^2}}dx.$$

Noting that

$$\begin{aligned}x - \frac{(x-\mu)^2}{2\sigma^2} &= \frac{2x\sigma^2}{2\sigma^2} - \frac{x^2 - 2\mu x + \mu^2}{2\sigma^2} = -\frac{x^2 - 2(\mu+\sigma^2)x + \mu^2}{2\sigma^2}\\ &= -\frac{x^2 - 2(\mu+\sigma^2)x + (\mu+\sigma^2)^2 - 2\mu\sigma^2 - \sigma^4}{2\sigma^2}\\ &= -\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2} + \mu + \frac{\sigma^2}{2},\end{aligned}$$

we obtain, by using (6) with $Y \sim N(\mu+\sigma^2, \sigma^2)$,

$$\begin{aligned}\mathbb{E}[e^X \mathbb{1}_{\{X \geq c\}}] &= e^{\mu+\frac{\sigma^2}{2}}\int_c^\infty \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-(\mu-\sigma^2))^2}{2\sigma^2}}dx = e^{\mu+\frac{\sigma^2}{2}}\int_c^\infty f_{\mu+\sigma^2,\sigma^2}(x)dx\\ &= e^{\mu+\frac{\sigma^2}{2}}\mathbb{P}(Y \geq c) = e^{\mu+\frac{\sigma^2}{2}}\Phi\Big(-\frac{c-(\mu+\sigma^2)}{\sigma}\Big).\end{aligned}$$

**Siehe nächstes Blatt!**

setting $X = \log L(T_{m-1}, T_{m-1})$, we obtain

$$
\begin{aligned}
&\mathrm{Cpl}(t; T_{m-1}, T_m) = P(t, T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}[\delta(L(T_{m-1}, T_{m-1}) - \varkappa)^+ \,|\, \mathcal{F}_t] \\
&= \delta P(t, T_m)\big(\mathbb{E}_{\mathbb{Q}^{T_m}}[e^X \mathbb{1}_{\{X \geq \log \varkappa\}} \,|\, \mathcal{F}_t] - \varkappa \mathbb{Q}^{T_m}(X \geq \log \varkappa \,|\, \mathcal{F}_t)\big) \\
&= \delta P(t, T_m)\left[e^{\mu + \frac{\sigma^2}{2}} \Phi\left( - \frac{\log \varkappa - (\mu + \sigma^2)}{\sigma}\right) - \varkappa \Phi\left( - \frac{\log \varkappa - \mu}{\sigma}\right)\right] \\
&= \delta P(t, T_m)\big(L(t, T_{m-1})\Phi(d_1(m; t)) - \varkappa \Phi(d_2(m; t))\big)
\end{aligned}
$$

with

$$
d_{1,2}(m; t) = \frac{\log\left(\frac{L(t, T_{m-1})}{\varkappa}\right) \pm \frac{1}{2}\int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds}{\left(\int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds\right)^{1/2}}.
$$

## 5. Matlab File

```matlab
function cap = capvasicekatm(b,beta,sigma,r0,T0,delta,Tcap)
% In this exercise (Ex 6-3) we compute the cap price in the vasicek model
% dr(t)= (b+ beta r(t))dt+ sigma dW^*(t)


%% Bond Price at time 0 (using the affine structure of Vasicek model Ex 4-3)
funcA= @(T) sigma^2*(4*exp(beta*T)-exp(2*beta*T)-2*beta*T-3)/(4*beta^3)+b*(exp(beta*T)-1-beta*T)/(beta^2);
funcB= @(T) (exp(beta*T)-1)/beta;
bondprice= @(T) exp(-funcA(T)-funcB(T)*r0);

%% Put price
% We first use Ex5-2) to find \sigma_{T,S}
% sigmap=\int_0^T \sigma^2_{T,S}(s)ds = \int_0^T (\int_T^S (\sigma exp(beta(u-s))) du)^2 ds
%sigmap = @(T,S) (4*exp(beta*S)*sigma^2*sinh(1/2*beta*(S-T))^2*sinh(beta*T))/(beta^3);
sigmap=@(T,S) sigma^2/(2*beta^3)*((exp(beta*(S-T))-1)^2*(exp(2*beta*T)-1));
d1 = @(T,S,K) (log(bondprice(S)/(K*bondprice(T)))+1/2*sigmap(T,S))/sqrt(sigmap(T,S));
d2 = @(T,S,K) (log(bondprice(S)/(K*bondprice(T)))-1/2*sigmap(T,S))/sqrt(sigmap(T,S));

```

**Bitte wenden!**

```matlab
putprice = @(T,S,K) K*bondprice(T)*normcdf(-d2(T,S,K))-bondprice(S)*normcdf(-d1(T,S,K));

%% ATM cap, cap rate = swap rate
tenor = T0:delta:Tcap;
kappa = (bondprice(T0)-bondprice(Tcap))/(delta*sum(bondprice(tenor(2:end))));


% EX 1-2) relation between CF of the caplet and put option
strikeprice = 1/(1+delta*kappa);
caplet = zeros(1,length(tenor)-1);
for j = 1:(length(tenor)-1)
    caplet(j)= putprice(tenor(j),tenor(j+1),strikeprice);
end
cap=sum(caplet)/strikeprice;
end
```

## 6. Matlab File

```matlab
function [sigmastar,value]=fittedsigma
% Ex 6-3b) find the optimal sigma_* which minimizes the L^2 error of
% observed ATM cap price  and calibrated Vasicek bond price

% lower bound for sigma_*
tol=10^(-3);

% find the minimal value
[sigmastar,value]= fminbnd(@(sigma) temp(sigma),tol,1);
end


function [error]=temp(sigma)
%parameters
b=0.0774;
beta=-0.86;
r0=0.08;

```

**Siehe nächstes Blatt!**

```
% observed ATM cap prices
P = [0.00215686,0.00567477,0.00907115,0.0121906,0.01503,0.017613,0.0199
% associated maturities
T =[1,2,3,4,5,6,7,8,10,12,15,20,30];

CVA= zeros(1,length(P));
for i=1:length(P)
    % computed ATM cap price
CVA(i) = capvasicekatm(b,beta,sigma,r0,1/4,1/4,T(i));
end

% L^2 norm of the error
error = sum((CVA-P).^2);
end
```

**7. Matlab File**

```
function [valueblack,valuemc]= liborcaplet
% In this exercise we compute the caplet price at time 0 with reset date
% T_{m-1}=(m-1)*T with cap rate kappa and settlement date T_m = m*T in a LIBOR market model
% with dL(t,T_m) = L(t,T_m) lambda* W^{T_{m+1}}_t

delta=1/4;
m=7;
lambda=0.1;
kappa=0.01;

% L(0,T_{m-1})
liborTmm1=0.09;
% P(0,T_m)
pTm= 0.85;
% number of Monte-Carlo simulations
N=10^7;
% Time T_{m-1}
Tmm1=(m-1)*delta;

%% Black's formula
```

**Bitte wenden!**

```
d1= (log(liborTmm1/kappa)+1/2*lambda^2*Tmm1)/(sqrt(lambda^2*Tmm1));
d2= (log(liborTmm1/kappa)-1/2*lambda^2*Tmm1)/(sqrt(lambda^2*Tmm1));

valueblack= delta*pTm*(liborTmm1*normcdf(d1)-kappa*normcdf(d2));


%% Monte-Carlo simulation
% mean of log(L(T_{m-1},T_{m-1})
mu = log(liborTmm1)-1/2*lambda^2*(Tmm1);
% std of log(L(T_{m-1},T_{m-1})
sigma = sqrt(lambda^2*(Tmm1));
% generate log(L(T_{m-1},T_{m-1})
logofLtmm1= mu+sigma*randn(1,N);
% evaluation
valuemc= pTm*mean(delta*subplus((exp(logofLtmm1)-kappa)));
end
```