Interest Rate Theory
Solution Sheet 7

1. **a)** First we note that
$$\mathbb{P}\left[V(t) = X\right] = 0$$

for both models. For the Merton Model, inserting the definition of V and using the independent increment property of Brownian motion yields

$$
\begin{aligned}
\mathbb{P}\left[V(T) < X | \mathcal{F}_t\right] &= \mathbb{P}\left[V(t) e^{(\mu - \sigma^2/2)(T-t) + \sigma W(T) - \sigma W(t)} < X | \mathcal{F}_t\right] \\
&= \mathbb{P}\left[\sigma(W(T) - W(t)) < \log(X/V(t)) - (\mu - \sigma^2/2)(T - t) | \mathcal{F}_t\right] \\
&= \mathbb{P}\left[\sigma\sqrt{T-t}Z < \log(X/V(t)) - (\mu - \sigma^2/2)(T - t)\right],
\end{aligned}
$$

where Z denotes a standard normal random variable on $\mathbb{R}$. Hence,

$$
\mathbb{P}\left[V(T) < X | \mathcal{F}_t\right] = \Phi\left(\frac{\log(X/V(t)) - (\mu - \sigma^2/2)(T - t)}{\sigma\sqrt{T-t}}\right) \quad (1)
$$

$$
\overset{T\downarrow t}{\Rightarrow} \begin{cases} \Phi(\infty) = 1, \text{ if } V(t) < X, \\ \Phi(-\infty) = 0, \text{ if } V(t) > X, \end{cases}
$$

where Φ denotes the cumulative distribution function of a standard normal random variable.

For the Zhou model we know from the lecture that the conditional default probability can be written as:

$$
\begin{aligned}
p^D(t, T) &= \mathbb{P}\left[V(T) < X | \mathcal{F}_t\right] \\
&= \sum_{n=0}^{\infty} \Phi\left(\frac{\log(X/V(t)) - mn - (\mu - \sigma^2/2)(T - t)}{\sqrt{n\varrho^2 + \sigma^2(T - t)}}\right) \\
&\quad \times e^{-\lambda(T-t)}\frac{(\lambda(T - t))^n}{n!}. \quad (2)
\end{aligned}
$$

Since the sum uniform convergent on $[t, T]$ (as $0 \leq \Phi(x) \leq 1$ for all x and the sum is absolute convergent) we can pull the limit of $T \downarrow t$ inside the sum and

Bitte wenden!

obtain

$$\lim_{T\downarrow t} p^D(t,T) \;=\; \lim_{T\downarrow t} \Phi\left(\frac{\log\left(X/V(t)\right) - (\mu - \sigma^2/2)(T-t)}{\sqrt{\sigma^2(T-t)}}\right)$$

$$=\; \begin{cases} \Phi(\infty) = 1, \text{ if } V(t) < X, \\ \Phi(-\infty) = 0, \text{ if } V(t) > X. \end{cases}$$

b) Using the formula (1) we get

$$\lim_{T\downarrow t} \partial_T^+ p^D(t,T) \;=\; \lim_{T\downarrow t} \partial_T^+ \Phi\left(\frac{\log(X/V(t)) - (\mu - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$=\; \lim_{T\downarrow t} \varphi\left(\frac{\log(X/V(t)) - (\mu - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$\times\; \frac{-(\mu - \sigma^2/2)\sigma\sqrt{T-t} - (\log(X/V(t)) + (\mu - \sigma^2/2)(T-t))\frac{1}{2\sigma\sqrt{T-t}}}{\sigma^2(T-t)}$$

$$=\; 0,$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ denotes the probability density of the standard normal distribution.

c) Similarly, using the formula (2) we have

$$\lim_{T\downarrow t} \partial_T^+ p^D(t,T) \;=\; \lim_{T\downarrow t} \partial_T^+ \sum_{n=0}^{\infty} \Phi\left(\frac{\log\left(X/V(t)\right) - mn - (\mu - \sigma^2/2)(T-t)}{\sqrt{n\varrho^2 + \sigma^2(T-t)}}\right)$$

$$\times e^{-\lambda(T-t)}\frac{(\lambda(T-t))^n}{n!}.$$

Again, since the sum is absolute convergent we may interchange the partial derivative with the sum:

$$\lim_{T\downarrow t} \partial_T^+ p^D(t,T) \;=\; \lim_{T\downarrow t} \sum_{n=0}^{\infty} \partial_T^+ \underbrace{\Phi\left(\frac{\log\left(X/V(t)\right) - mn - (\mu - \sigma^2/2)(T-t)}{\sqrt{n\varrho^2 + \sigma^2(T-t)}}\right)}_{=:f(T,n)}$$

$$\times e^{-\lambda(T-t)}\frac{(\lambda(T-t))^n}{n!}$$

$$=\; \lim_{T\downarrow t} \sum_{n=0}^{\infty} \left[\varphi(f(T,n))\partial_T f(T,n) e^{-\lambda(T-t)}\frac{(\lambda(T-t))^n}{n!}\right.$$

$$+\Phi(f(T,n))(-\lambda)e^{-\lambda(T-t)}\frac{(\lambda(T-t))^n}{n!}$$

$$\left.+\Phi(f(T,n))e^{-\lambda(T-t)}\frac{(\lambda(T-t))^{n-1}}{(n-1)!}\lambda\right].$$

Since φ decays exponentially fast to zero the first summand can be neglected:

$$
\begin{aligned}
\lim_{T\downarrow t} \partial_T^+ p_d(t,T) &= -\lambda\Phi(f(t,0)) + \Phi(f(t,1))\lambda \\
&= -\lambda\mathbb{I}_{\{V(t)<X\}} + \Phi(d)\lambda \\
&= -\lambda\mathbb{I}_{\{V(t)<X\}} + \Phi(d)\lambda\mathbb{I}_{\{V(t)<X\}} + \Phi(d)\lambda\mathbb{I}_{\{V(t)\geq X\}} \\
&= \lambda\Phi(d)\mathbb{I}_{\{V(t)\geq X\}} + \lambda\Phi(-d)\mathbb{I}_{\{V(t)<X\}}.
\end{aligned}
$$

2. From the lecture we know that the conditional default probability

$$
\mathbb{Q}\left[\tau > T | \mathcal{F}_T\right] = e^{-\int_0^T \lambda(s)ds}. \tag{3}
$$

Hence,

$$
\mathbb{Q}\left[\tau > T\right] = \mathbb{E}\left[\mathbb{Q}\left[\tau > T | \mathcal{F}_T\right]\right] = \mathbb{E}\left[e^{-\int_0^T \lambda(s)ds}\right].
$$

Therefore, to conclude we have to check that

$$
\mathbb{E}\left[e^{-\int_0^T \lambda(s)ds}\right] = e^{-A(T)-B(T)\lambda(0)}.
$$

Since the process λ is assumed to be follow a CIR process we can apply the same ideas as in the affine short rate chapter to compute $\mathbb{E}\left[e^{-\int_0^T \lambda(s)ds}\right]$. That is, we solve the corresponding ODEs (cf. Ex 4-3):

$$
\begin{aligned}
\partial_t B(t,T) &= \frac{\sigma^2}{2} B(t,T)^2 - \beta B(t,T) - 1, \quad B(T,T) = 0, \\
\partial_t A(t,T) &= -bB(t,T), \quad A(T,T) = 0.
\end{aligned}
$$

Indeed, the functions A and B given on the exercise sheet satisfy the above ODEs which can be checked by insertion.

3. **a)** In the following expectations are taken under the measure $\mathbb{Q}$. Using Tower property we have

$$
\begin{aligned}
C^0(T) &= \mathbb{E}\left[e^{-\int_0^T r(s)ds}\mathbb{I}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-\int_0^T r(s)ds}\mathbb{I}_{\{\tau>T\}} | \mathcal{F}_T\right]\right] \\
&= \mathbb{E}\left[e^{-\int_0^T r(s)ds}\mathbb{E}\left[\mathbb{I}_{\{\tau>T\}} | \mathcal{F}_T\right]\right] \\
&= \mathbb{E}\left[e^{-\int_0^T (r(s)+\lambda(s))ds}\right] \\
&= \mathbb{E}\left[e^{-\int_0^T (c_0+(1+c_1)r(s))ds}\right].
\end{aligned}
$$

Following the same lines of reasoning as in the previous exercise (adapt the proof of Lemma 5.1 and Proposition 5.2 in Filipovic's book), the following two ODEs have to be solved:

$$\partial_t B(t,T) = \frac{\sigma^2}{2} B(t,T)^2 - \beta B(t,T) - (1+c_1), \quad B(T,T) = 0,$$

$$\partial_t A(t,T) = -bB(t,T) - c_0, \quad A(T,T) = 0.$$

Again, the functions given on the exercise sheet satisfy the above ODEs.

b) To verify the claim we first note that

$$\mathbb{I}_{\{\tau>T\}} + \delta\mathbb{I}_{\{\tau\leq T\}} = (1-\delta)\mathbb{I}_{\{\tau>T\}} + \delta.$$

Therefore, part a) yields

$$C(T) = (1-\delta)C^0(T) + \delta P(0,T).$$

c) We note that the zero-coupon bond price with partial recovery at default is given by

$$C^D(T) = C^0(T) + \delta\Pi(T),$$

where

$$\Pi(T) = \mathbb{E}\left[e^{-\int_0^\tau r(s)ds} \mathbb{I}_{\{0<\tau\leq T\}} \right]$$

is the unit price of the recovery at default given $0 < \tau \leq T$. Again with Tower property and using the exponential distribution of τ we get

$$\begin{aligned}
\Pi(T) &= \mathbb{E}\left[\mathbb{E}\left[e^{-\int_0^\tau r(s)ds} \mathbb{I}_{\{0<\tau\leq T\}} \middle| \mathcal{F}_T \right] \right] \\
&= \mathbb{E}\left[\int_0^T \left(e^{-\int_0^u r(s)ds} \lambda(u) e^{-\int_0^u \lambda(s)ds} \right) du \right] \\
&= \int_0^T \mathbb{E}\left[\lambda(u) e^{-\int_0^u (r(s)+\lambda(s))ds} \right] du \\
&= \int_0^T \mathbb{E}\left[\lambda(u) e^{-\int_0^u (c_0+(1+c_1)r(s))ds} \right] du.
\end{aligned}$$

With dominated convergence (since r is positive) we have

$$
\begin{aligned}
\frac{c_1}{1+c_1}\frac{d}{du}\mathbb{E}\left[e^{-\int_0^u(r(s)+\lambda(s))ds}\right] &= \frac{c_1}{1+c_1}\mathbb{E}\left[\frac{d}{du}e^{-\int_0^u(r(s)+\lambda(s))ds}\right]\\
&= \frac{c_1}{1+c_1}\mathbb{E}\left[-(c_0+(1+c_1)r(u))e^{-\int_0^u(r(s)+\lambda(s))ds}\right]\\
&= -\frac{c_0c_1}{1+c_1}\mathbb{E}\left[e^{-\int_0^u(r(s)+\lambda(s))ds}\right]-\mathbb{E}\left[c_1r(u)e^{-\int_0^u(r(s)+\lambda(s))ds}\right]\\
&= \left(-\frac{c_0c_1}{1+c_1}+c_0\right)\mathbb{E}\left[e^{-\int_0^u(r(s)+\lambda(s))ds}\right]\\
&\quad -\mathbb{E}\left[\lambda(u)e^{-\int_0^u(r(s)+\lambda(s))ds}\right]\\
&= \left(\frac{c_0}{1+c_1}\right)\mathbb{E}\left[e^{-\int_0^u(r(s)+\lambda(s))ds}\right]\\
&\quad -\mathbb{E}\left[\lambda(u)e^{-\int_0^u(r(s)+\lambda(s))ds}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Pi(T) &= \left(\int_0^T \frac{c_0}{1+c_1}e^{-A(u)-B(u)r(0)}du - \frac{c_1}{1+c_1}\int_0^T \frac{d}{du}e^{-A(u)-B(u)r(0)}du\right)\\
&= \left(\int_0^T \frac{c_0}{1+c_1}e^{-A(u)-B(u)r(0)}du - \frac{c_1}{1+c_1}\left(e^{-A(T)-B(T)r(0)}-1\right)\right).
\end{aligned}
$$

4. a) With $C_1 = C_2 = \Phi^{-1}(p)$ we have

$$
\begin{aligned}
p(X) &= \mathbb{P}\left[Y_i = 1|X\right]\\
&= \mathbb{P}\left[W_i(1) < C_i|X\right]\\
&= \mathbb{P}\left[\sqrt{\varrho}X + \sqrt{1-\varrho}Z_i < \Phi^{-1}(p)\right]\\
&= \Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\varrho}X}{\sqrt{1-\varrho}}\right).
\end{aligned}
$$

Conditionally on X the Y_i are independent Bernoulli variables with success probability $p(X)$, therefore

$$
\mathbb{P}\left[Y_1 = y_1, Y_2 = y_2\right] = \int_{\mathbb{R}} p(x)^{y_1+y_2}(1-p(x))^{2-y_1-y_2}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx.
$$

In the special case $y_1 = y_2 = 1$ we have

$$
\mathbb{P}\left[Y_1 = 1, Y_2 = 1\right] = \int_{\mathbb{R}} p(x)^2\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = \int_{\mathbb{R}} \Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\varrho}x}{\sqrt{1-\varrho}}\right)^2\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx.
$$

Bitte wenden!

b) Conditionally on X, the random variable L_2 has a Binomial distribution with parameter vector $(2, p(X))$. Therefore

$$\mathbb{P}[L_2 \leq k] = \sum_{i=0}^{k} \binom{n}{i} \int_{\mathbb{R}} p(x)^i (1 - p(x))^{2-i} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

c) Matlab File

```matlab
1  function empiricaldf
2  % In this exercise we compute the emipirical
       distribution function of L_2
3  % in a homogeneous Bernoulli mixture model
4
5  % set of parameters
6  mu=0.01;
7  sigma=0.2;
8  rho=0.5;
9  V0=100;
10 D=90;
11
12 %% compute theoretical cdf
13 % standardized face value of the debt
14 C= (log(D/V0)-(mu-sigma^2/2))/sigma;
15
16 % compute p(X)
17 jointdefault = @(x) normcdf((C-sqrt(rho)*x)/sqrt(1-
       rho));
18
19 % compute the theoretical pdf
20 integrand = @(x,i) jointdefault(x).^i.*(1-
       jointdefault(x)).^(2-i).*normpdf(x);
21 temp  = @(i) nchoosek(2,i).* integral(@(x) integrand(
       x,i),-Inf,Inf);
22
23
24 % possible values for i
25 grid=[0,1,2];
26 values=cumsum([temp(0),temp(1),temp(2)]);
27
28
29
30 %% compute empirical cdf
31 N= 10^4;
```

```matlab
32 % simulate X
33 draw = randn(1,N);
34
35 % simulate Y_1 and Y_2
36 y1draw = binornd(1, jointdefault(draw));
37 y2draw = binornd(1, jointdefault(draw));
38 % compute L= Y_1+Y_2
39 Ldraw = y1draw+y2draw;
40
41 % compute the empiricial distribution function
42 empirical=ecdf(Ldraw);
43
44 % plot the theoretical cdf and the empirical cdf
45 figure(1);
46 stairs(grid, values, 'r*-')
47 hold on;
48 stairs(grid, empirical(2:end), 'g*-')
49 hold off;
50 end
```