ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Exercise Sheet 3

1. The goal of the exercise is to show that completeness does not imply the absence of arbitrage. In this exercise we use the following definition of admissibility: φ is an *admissible* startegy, if it is self financing and the value process associated to φ is bounded from below, i.e., there exists $a > 0$ such that $V^\varphi \geq -a$ a.s..

a) Let $\mathcal{F} = \{\Omega, \emptyset\}$ be the trivial $\sigma-$algebra, and consider the deterministic financial market model with zero interest rate, $S_0 \equiv 1$, and $n = 1$ additional asset $S_1(t) = 100 + t$. Show that this model is complete but not free of arbitrage.

b) Is this also possible in a non-deterministic financial market?

2. Let $W := (W(t))_{t\in[0,T]}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by W. Consider the Bachelier-model described by a market (S_0, S_1) with

$$\begin{aligned} &S_0(t) \equiv 1, \\ &S_1(t) = S_1(0)(1 + \mu t + \sigma W(t)) \end{aligned}$$

for constants $\mu \in \mathbb{R}$ and $S_1(0), \sigma > 0$.

a) Derive an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ and show that it is unique.

b) Show that the price $C_t(T, S_1(t), K, \sigma) = \mathbb{E}_\mathbb{Q}[(S_1(T) - K)^+|\mathcal{F}_t]$ at time $t \leq T$ of a European call option with maturity T and strike K is given by the formula

$$\begin{aligned} C_t(T, S_1(t), K, \sigma) =& S_1(0)\sigma\sqrt{T-t}\phi\left(\frac{S_1(t) - K}{S_1(0)\sigma\sqrt{T-t}}\right) \\ &+ (S_1(t) - K)\Phi\left(\frac{S_1(t) - K}{S_1(0)\sigma\sqrt{T-t}}\right), \end{aligned}$$

where ϕ stands for the standard Gaussian density function and Φ denotes the standard Gaussian cumulative distribution function:

$$\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad \Phi(x) := \int_{-\infty}^{x} \phi(z) dz, \quad x \in \mathbb{R}.$$

Bitte wenden!

3. We consider CIR process as defined in Ex 2-4, i.e.,

$$dY(t) = \lambda(\nu - Y(t))dt + \sigma\sqrt{Y(t)}dW_t, \quad Y_0 = y, \tag{1}$$

where ν and $\lambda, \sigma > 0$ are real constants and W is standard Brownian motion.

a) Show that for $(2 \leq) d := 4\lambda\nu/\sigma^2 \in \mathbb{N}$ the CIR process can represented in terms of a d-dimensional Ornstein-Uhlenbeck process. That is, there exists a standard Brownian motion W and a d-dimensional OU process[1] X starting at $(\sqrt{y}, 0, \cdots, 0)$ such that

$$Y_t = \|X_t\|^2_{\mathbb{R}^d}$$

fulfills the SDE (1), where the process $X = (X_1, \cdots, X_d)$ satisfies

$$dX_i(t) = -\frac{1}{2}\lambda X_i(t)dt + \frac{\sigma}{2}dB_i(t)$$

with a d-dimensional Brownian motion $B = (B_1, \cdots, B_d)$.
Hint: Apply Itô's Formula and Lévy's characterization Theorem to find the Brownian motion W.

b) For $s < t$ it can be shown that condition on Y_s the random variable $c_t Y_t$ follows a non-central χ^2- distribution with degree of freedom parameter $4\lambda\nu/\sigma^2$ and scale parameter $c_t Y_s e^{-\lambda(t-s)}$, where $c_t = 4\lambda/(\sigma^2(1 - e^{-\lambda(t-s)}))$. Use this transition property of the CIR process to simulate 10 sample paths and repeat the Monte Carlo simulations from Ex 2-4. That is,

- Simulate 10 sample paths using an equidistant grid on $[0, 1]$ with grid size = 10^{-3} and parameters $\lambda = 1$, $\nu = 1.2$, $\sigma = 0.3$ and $y = 1$.
- Compute $\mathbb{E}[Y_1]$, $\mathbb{E}[Y_1^2]$ and $\mathbb{E}[Y_1^+]$ using Monte Carlo simulation with $N = 10^4$.

Hint: use the command $ncx2rnd$ in Matlab

4. (Bayes' rule) Let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure and denote its density process by $D(t) = d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t}$. Let X be an $\mathcal{F}_T$-measurable random variable satisfying $\mathbb{E}_{\mathbb{Q}}[|X|] < \infty$.

a) Show that we have the Bayes' rule

$$\mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_t] = \frac{\mathbb{E}[XD(T) \mid \mathcal{F}_t]}{D(t)}, \quad t \leq T.$$

b) As an application, show that an adapted process M is a $\mathbb{Q}$-martingale if and only if DM is a $\mathbb{P}$-martingale.

[1] That is, $X = (X_1, \cdots, X_d)$ where the X_i's are independent OU processes.

Siehe nächstes Blatt!

5. **Matlab Exercise** Let us model the market according to the Bachelier-model as described in Ex3-2. Derive an expression for the call price of a European call option

$$C_0(T, S_1(0), K, \sigma)$$

which is *at-the-money*, i.e. $S_1(0) = K$ and write a function $impvol(C, K, T)$ in Matlab, which determines the implied volatility σ for a given strike K, maturity T, and value C of an at-the-money call option at time $t = 0$.

The S&P 500 Index closed at $S_1(0) = 1125$ at an appointed date in April 2002. The quotes for at-the-money European call options with corresponding maturities were the following:

quoted call price $C_0(T)$	Maturity date
20.20	May 2002
30.70	June 2002
51.00	Sep. 2002
66.90	Dec. 2002
81.70	March 2003
97.00	June 2003

Test your function for the above values and plot the implied volatility σ against the time to Maturity T.