ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Exercise Sheet 4

1. Let us consider the general short rate model introduced in the lecture. That is, we assume,

(i) the short rate follows an Itô process

$$dr(t) = b(t)dt + \sigma(t)dW_t$$

determining the money-market account $B(t) = \exp\left(\int_0^t r(s)ds\right)$

(ii) no arbitrage: there exists an EMM $\mathbb{Q}$ of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\gamma \bullet W)_\infty$$

such that the discounted bond prices $P(t,T)/B(t), t \leq T$ are $\mathbb{Q}-$ martingales and $P(T,T) = 1$ for all $T > 0$.

Under the above assumptions show that

a) the process r satisfies under $\mathbb{Q}$

$$dr(t) = (b(t) + \sigma(t)\gamma(t))\, dt + \sigma(t)dW^{\mathbb{Q}}(t),$$

where $W^{\mathbb{Q}}$ denotes a $\mathbb{Q}-$ Brownian motion.

b) If the filtration $(\mathcal{F}_t)$ is generated by the Brownian motion W, for any $T > 0$ there exists a process $v(t,T) \in \mathcal{L}$ such that

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + v(t,T)dW^{\mathbb{Q}}(t).$$

c) Conclude that

$$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}\left(v(\cdot,T) \bullet W^{\mathbb{Q}}\right)_t.$$

Bitte wenden!

2. Compute directly the price at time t of a zero coupon bond with maturity date T in the Vasiček model

$$P(t,T) = \mathbb{E}\Big[e^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big], \tag{1}$$

where the short rate $(r(t))_{t\geq 0}$ is modeled by the OU-process[1]

$$r(t) = r_0 e^{\beta t} + \frac{b}{\beta}(e^{\beta t} - 1) + \sigma e^{\beta t}\int_0^t e^{-\beta s}dW_s, \quad t \geq 0, \tag{2}$$

for constants $b, \sigma \in \mathbb{R}$, $\beta < 0$ and $r_0 \in \mathbb{R}$.

3. Consider again the Vasiček model as in Ex-4-2.

a) Determine *term-structure equation* associated to it, i.e, find the partial differential equation such that the process defined by

$$M(t) = \mathbb{E}\left[e^{-\int_0^T r(s)ds}\Big|\mathcal{F}_t\right] = F(t, r(t); T)e^{-\int_0^t r(s)ds}, \quad 0 \leq t \leq T \tag{3}$$

is a local martingale.

b) Assuming the process M defined in (3) is a true martingale, solve the term-structure equation for $F(T, r(T); T) = 1$ associated to the Vasiček model. Moreover, determine the associated bond prices by using

$$P(t,T) = F(t, r(t); T).$$

Compare your results with the bond prices (1) obtained in Ex 4-2.

c) Show that the process M is indeed a true martingale.

4. Matlab-Exercise The goal of this exercise is numerically compute the time zero bond price in the Vasiček model

$$P(0,T) = \mathbb{E}\left[e^{-\int_0^T r_s ds}\right]$$

by using three different methods.

[1]Recall from the Ex 2-3 that the Ornstein-Uhlenbeck process r satisfies

$$dr(t) = -\beta\left(-\frac{b}{\beta} - r(t)\right)dt + \sigma dW(t), \quad r(0) = r_0.$$

Siehe nächstes Blatt!

a) *Analytical Approach:* In Ex 4-3 we have seen that $P(0,T)$ can written as

$$P(0,T) = F(0, r_0; T) = \exp(-A(T) - B(T)r_0),$$

with

$$A(T) = \frac{\sigma^2(4e^{\beta T} - e^{2\beta T} - 2\beta T - 3)}{4\beta^3} + b\frac{e^{\beta T} - 1 - \beta T}{\beta^2},$$
$$B(T) = \frac{1}{\beta}\big(e^{\beta T} - 1\big).$$

b) *Monte Carlo Approach:* In Ex 4-2 it was shown that the integral $\int_0^T r_s ds$ is normally distributed with mean μ_0 and variance Σ_0^2 where

$$\mu_0 = \frac{r_0}{\beta}(e^{\beta T} - 1) + \frac{b}{\beta^2}(e^{\beta T} - 1 - \beta T),$$
$$\Sigma_0^2 = \frac{\sigma^2(-4e^{\beta T} + e^{2\beta T} + 2\beta T + 3)}{2\beta^3}.$$

Recall that the essential idea of Monte Carlo simulation is that – by the law of large numbers – for large $N \in \mathbb{N}$ and an i.i.d. sequence $X_1, \ldots, X_N$ having the distribution of $e^{-\int_0^T r_s ds}$ we have

$$P(0,T) \approx \frac{1}{N}\sum_{k=1}^{N} X_k.$$

c) *Euler-Maruyama Approach:* An alternative method is to simulate the short rate r explicitly using Euler-Maruyama scheme and apply the trapezoidal rule to compute the integral $\int_0^T r_s ds$, i.e.,

$$\int_0^T r_s ds \approx \sum_{i=1}^{M}(t_i - t_{i-1})\frac{r_{t_{i-1}} + r_{t_i}}{2},$$

where we consider the equidistant decomposition $\{0 = t_0 < \ldots < t_M = T\}$ of the interval $[0,T]$ given by

$$t_i := \frac{i}{M}T, \quad i = 0, \ldots, M.$$

Finally, we again use Monte-Carlo simulation to approximate the expectation. Implement these three methods in Matlab and compare your results for the following set of parameters

$$b = 0.08, \beta = -0.86, \sigma = 0.04, r_0 = 0.08, T = 10, M = 10^3, N = 10^5.$$

Hint: The command for trapezoidal rule in Matlab is *trapz*.