ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Exercise Sheet 5

1. Calculate $\mathbb{E}[r(t)]$ and determine its behavior as $t \to \infty$ for the following processes

a) CIR short rate model:

$$r(t) = r_0 + \int_0^t (a - br(s))ds + \sigma \int_0^t \sqrt{r(s)}dW(s),$$

with initial value $r_0 > 0$ and constants $a, b, \sigma > 0$ such that $2a \geq \sigma^2$.

b) Dothan short rate model:

$$r(t) = r_0 + \beta \int_0^t r(s)ds + \sigma \int_0^t r(s)dW(s),$$

with initial value $r_0 > 0$ and constants $\beta \in \mathbb{R}, \sigma > 0$.

2. We consider the *Hull-White* extension of the Vasiček short-rate dynamics under an EMM $\mathbb{Q} \sim \mathbb{P}$:

$$r(t) = r_0 + \int_0^t (b(s) + \beta r(s))ds + \sigma W^*(t),$$

where W^* is a standard, real-value Brownian motion, with initial value $r_0 \in \mathbb{R}$, with a deterministic, continuous function $b : \mathbb{R}_+ \mapsto \mathbb{R}$ and constants $\beta < 0$ and $\sigma > 0$. Let $T \mapsto P^*(0,T)$ be a C^2- curve of initial bond prices. The goal of this exercise is to find the function

$$b : \mathbb{R}_+ \mapsto \mathbb{R}$$

such that the Hull-White extension of the Vasiček model fits this initial bond curve, i.e.,

$$P(0,T) = P^*(0,T), \quad \text{for all} \quad T \geq 0.$$

Bitte wenden!

a) Find the corresponding HJM forward rate dynamics

$$f(t,T) = f(0,T) + \int_0^t \underbrace{\sigma(s,T)\int_s^T \sigma(s,u)du}_{=:\alpha(s,T)}\, ds + \int_0^t \sigma(s,T)dW^*(s)$$

and determine the function b using the HJB drift condition.

b) How would one proceed without the HJM framework to fit the model to the initial data?

3. The goal of this exercise is to show that parallel shifts of the forward curve creates arbitrage. Consider the one-period model for the forward-curve

$$\begin{aligned} f(0,T) &= 0.04, \quad T \geq 0, \\ f(\omega,1,T) &= \begin{cases} 0.06, & T \geq 1, \omega = \omega_1, \\ 0.02, & T \geq 1, \omega = \omega_2, \end{cases} \end{aligned}$$

where $\Omega = \{\omega_1, \omega_2\}$ with $\mathbb{P}[\omega_i] > 0, i = 1, 2$.

a) Show that the matrix

$$M = \begin{pmatrix} P(0,1) & P(0,2) & P(0,3) \\ P(1,1)(\omega_1) & P(1,2)(\omega_1) & P(1,3)(\omega_1) \\ P(1,1)(\omega_2) & P(1,2)(\omega_2) & P(1,3)(\omega_2) \end{pmatrix}$$

is invertible.

b) Use (a) to find an arbitrage strategy with value process $V(0) = 0$ and $V(1)(\omega_1) = V(1)(\omega_2) = 1$.

4. Let the initial forward curve be described by $f(0,T) = h(T)$ for some deterministic, smooth funtion $T \mapsto h(T)$. Moreover, let Z be an Itô process with the dynamics

$$dZ(t) = b(t)dt + \varrho(t)dW^*(t), \qquad Z(0) = 0.$$

An HJM forward curve evolution by parallel shifts is described by

$$f(t,T) = h(T-t) + Z(t). \tag{1}$$

a) Show that the HJM drift condition implies $b(t) \equiv b$, $\varrho^2(t) \equiv a$, and

$$h(x) = -\frac{a}{2}x^2 + bx + c$$

for some constants $a \geq 0$, and $b, c \in \mathbb{R}$.

Siehe nächstes Blatt!

b) Identify the short-rate model determined by the forward curve (1).

c) Consider now a generic initial forward curve. Argue why the HJM drift condition excludes non-trivial forward curve evolution by parallel shifts.

5. Matlab Exercise: The goal of this exercise is numerically compute the time zero bond price

$$P(0,T) = \mathbb{E}\left[e^{-\int_0^T r_s ds}\right]$$

in the CIR resp. Dothan model from Ex 5-1 (use $M = 10^3$ discretization points and $N = 10^5$ sample size). Simulate the corresponding short rate processes with Euler-Maruyama scheme and use the trapezoidal rule to approximate the integral as in Ex 4-5. Recall in Ex 4-4 we have computed bond price in the Vasiček model. In order to compare the Vasiček bond price with the CIR resp. Dothan bond price we fix $T = 10$ and choose the other parameters such that expectation and variance of the r_T match, i.e.,

$$a = 0.0052, b = 0.0447, \sigma = 0.04, r_0 = 0.08$$

resp.

$$\beta = 0.0151, \sigma = 0.1011, r_0 = 0.08.$$