ETH Zürich HS 2014
D-MATH
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# Interest Rate Theory
## Exercise Sheet 6

**1.** The goal of this exercise is to verify the *Dybvig-Ingersoll-Ross* Theorem for some specific models.

**a)** Show that the Vasiček short-rate model

$$dr(t) = (b + \beta r(t))dt + \sigma dW^*(t)$$

admits a long rate $R_\infty(t) = \lim_{T\to\infty} \frac{1}{T-t}\int_t^T f(t,s)ds$ if $\beta < 0$. Conclude that

$$\lim_{x\to\infty} f(t, t+x) = -\left(\frac{b}{\beta} + \frac{\sigma^2}{2\beta^2}\right)$$

and verify that $R_\infty(t)$ is non-decreasing.

**b)** Show that the long rate $R_\infty(t)$ always exists in the CIR model

$$dr(t) = (b + \beta r(t))dt + \sigma\sqrt{r(t)}dW^*(t),$$

and verify that it is non-decreasing.

**c)** Determine the long rate $R_\infty(t)$ in a one-dimensional HJM model with volatility process given by

$$\sigma(t,T) = \frac{1}{(1+T-t)^{1/2}}.$$

Show that it is strictly increasing.

**2.** Consider a bond market where the interest rate dynamics are given by the Ho-Lee short-rate model

$$dr(t) = b(t)dt + \sigma dW^*(t).$$

**a)** Derive a formula for the price $c(0,T,S,K)$ of a European call option at time $t = 0$ with strike price $K$ and exercise date $T$ on an underlying S-bond, where $T < S$.

**Bitte wenden!**

**b)** Derive the corresponding formula for European put option prices $p(0, T, S, K)$ in the above setting.

**3. Matlab Exercise** Consider the Vasiček short-rate model

$$dr(t) = (b + \beta r(t))dt + \sigma dW^*(t).$$

In this setting, the price at time $t = 0$ of a European put option $p(0, T, S, K)$ on a S-bond with expire date $T < S$ and strike price $K$ is

$$p(0, T, S, K) = KP(0, T)\Phi(-d_2) - P(0, S)\Phi(-d_1),$$

with

$$d_{1,2} = \frac{\log\left(\frac{P(0,S)}{KP(0,T)}\right) \pm \frac{1}{2}\int_0^T \sigma_{T,S}^2(s)ds}{\sqrt{\int_0^T \sigma_{T,S}^2(s)ds}},$$

where

$$\sigma_{T,S}(s) = \int_T^S \sigma(s, u)du.$$

**a)** Write a function *capvasicekatm(*$b, \beta, \sigma, r_0, T_0, \delta, T_{cap}$*)* which computes the price of ATM caps under the Vasiček short-rate model. Here, $T_0$ denotes the first reset date of the cap, $\delta = T_i - T_{i-1}$ and $T_{\text{cap}}$ is the maturity of the cap.

*Hint:* First identify $\sigma(s, u)$ using Ex 5-2. In a second step relate the cash flow of a caplet to the cash flow of the put option (cf. Ex 1-2).

**b)** For the following parameters

$$b = 0.0774, \beta = -0.86, r(0) = 0.08$$

write a function *fittedsigma* which minimizes the error between a given vector of the market prices $P$ and the price determined by Ex 6-3a) within the interval $\sigma \in [10^{-3}, 1]$. That is, the output $\sigma_*$ of the function *fittedsigma* is determined by

$$\sigma_* = argmin_{\sigma \in [10^{-3},1]} \sum_{i=1}^{k} (p_i - CVA_i(\sigma))^2,$$

where $P = (p_1, \cdots, p_k)$ is a given price vector and $CVA_i$ denotes the calculated cap prices from Ex 6-3a). Test your functions for the following set of prices

**Siehe nächstes Blatt!**

| Maturity of Cap | ATM cap price |
|---|---|
| 1 | 0.00215686 |
| 2 | 0.00567477 |
| 3 | 0.00907115 |
| 4 | 0.0121906 |
| 5 | 0.01503 |
| 6 | 0.017613 |
| 7 | 0.0199647 |
| 8 | 0.0221081 |
| 10 | 0.025847 |
| 12 | 0.028963 |
| 15 | 0.0326962 |
| 20 | 0.0370565 |
| 30 | 0.0416089 |

The maturities in the above table are given in years from today ($t_0 = 0$), the first reset date of each cap is $T_0 = 1/4$. All future dates $T_0 < \cdots < T_n$ are equidistant, i.e., $T_i - T_{i-1} \equiv 1/4$ for all $i = 1, \cdots, 119$. The maturity of the last cap is $T_{119} = 30$.

**4.** Let $\delta > 0$ and $T_m = m\delta$ with $m = 0, \ldots, M \in \mathbb{N}$. We assume that

- For every $m = 1, \ldots, M-1$ there exists a real-valued deterministic bounded measurable function $\lambda(t, T_m)$, $t \in [0, T_m]$.
- There is a positive, nonincreasing initial term structure

$$P(0, T_m), \quad m = 0, \ldots, M,$$

and hence nonnegative initial LIBOR rates

$$L(0, T_m) = \frac{1}{\delta}\left(\frac{P(0, T_m)}{P(0, T_{m+1})} - 1\right), \quad m = 0, \ldots, M-1.$$

We consider the LIBOR market model from the lecture, that is, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T_M]}, \mathbb{Q}^{T_M})$ there exist equivalent probability measures $(\mathbb{Q}^{T_m})_{m=1,\ldots,M-1}$ and $\mathbb{Q}^{T_m}$-Brownian motions $(W^{T_m})_{m=1,\ldots,M}$ such that for each $m = 0, \ldots, M-1$ (under the measure $\mathbb{Q}^{T_{m+1}}$) we have

$$L(t, T_m) = L(0, T_m) + \int_0^t L(s, T_m)\lambda(s, T_m)dW_s^{T_{m+1}}, \quad t \in [0, T_m].$$

Now, let $1 \le m \le M$ and $t \in [0, T_{m-1}]$. It can be shown[1] that for the time $t$ price of a caplet with reset date $T_{m-1}$, settlement date $T_m$ and strike rate $\varkappa$ is given by the

[1]Please verify this result if it is not clear to you.

**Bitte wenden!**

following Black-formula

$$\mathrm{Cpl}(t; T_{m-1}, T_m) = \delta P(t, T_m)(L(t, T_{m-1})\Phi(d_1(m; t)) - \varkappa\Phi(d_2(m; t))),$$

where we have set

$$d_{1,2}(m; t) = \frac{\log\left(\frac{L(t,T_{m-1})}{\varkappa}\right) \pm \frac{1}{2}\int_t^{T_{m-1}} \lambda^2(s, T)ds}{\left(\int_t^{T_{m-1}} \lambda^2(s, T_{m-1})ds\right)^{1/2}}.$$

Under the assumption that $\lambda(t, T_m) = \lambda$ is constant. Compute the caplet price at time $t = 0$

- by using Black's formula,
- by a Monte-Carlo algorithm (N=$10^6$).

You may test your results with the parameters

$$\delta = 1/4, \quad m = 7, \quad \lambda = 0.1, \quad \varkappa = 0.01,$$
$$L(0, T_{m-1}) = 0.09, \quad P(0, T_m) = 0.85, \quad N = 10^6.$$