ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Exercise Sheet 7

1. The aim of the exercise is to compute the instantaneous default intensity

$$\lim_{T\downarrow t} \partial_T^+ p^D(t,T)$$

for the structural credit-risk models introduced in the lecture.
Recall that in the Merton model the total market value of the company $V(t)$ is assumed to follow a geometric Brownian motion, i.e.,

$$dV(t)/V(t) = \mu dt + \sigma dW(t), \quad t \in [0,T].$$

Zhou extended Merton's model by using a jump diffusion, i.e.,

$$V(t) = V(0) + \int_0^t V(s)(\mu ds + \sigma dW(s)) + \sum_{j=1}^{N(t)} V(\tau_{j-})\left(e^{Z_j} - 1\right),$$

where $N(t)$ is a Poisson process with intensity λ and $Z_1, Z_2, \ldots$, is a sequence of i.i.d. $\mathcal{N}(m, \varrho^2)-$ distributed random variables, and $\tau_1, \tau_2, \ldots$ are the jump times of the process N. Moreover, it is assumed that W, N and Z_j are mutually independent.

a) Show that

$$\lim_{T\downarrow t} \mathbb{P}[V(T) < X|\mathcal{F}_t] = \lim_{T\downarrow t} p^D(t,T) = \mathbb{I}_{\{V(t)<X\}} \quad a.s.$$

for both models.

b) Show that Merton's default probability $p^D(t,T)$ satisfies

$$\lim_{T\downarrow t} \partial_T^+ p^D(t,T) = 0 \quad a.s.,$$

where $\Phi(x)$ denotes the cumulative distribution function of a standard normal random variable.

Bitte wenden!

c) Show that Zhou's default probability satisfies

$$\lim_{T \downarrow t} \partial_T^+ p^D(t,T) = \lambda \Phi(d) \mathbb{I}_{\{V(t) \geq X\}} - \lambda \Phi(-d) \mathbb{I}_{\{V(t) < X\}} \quad a.s.,$$

where

$$d = \frac{\log(X/V(t)) - m}{\varrho}.$$

2. Consider the reduced-form credit-risk models introduced in the lecture where default occurs at the first jump time τ of a Cox process[1], i.e., a generalization of a Poisson process where the time-dependent intensity $\lambda(t)$ is itself a stochastic process. In this exercise we use a CIR process to model λ, i.e.,

$$d\lambda(t) = (b + \beta\lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t), \quad \lambda(0) \geq 0,$$

where $b \geq 0, \beta \in \mathbb{R}, \sigma > 0, 2b \geq \sigma^2$, and W is a $(\mathbb{Q}, \mathcal{F})-$ Brownian motion. Show that the default probability is given by

$$\mathbb{Q}[\tau \leq T] = 1 - e^{-A(T) - B(T)\lambda(0)}$$

where

$$\begin{aligned} A(u) &= -\frac{2b}{\sigma^2} \log\left(\frac{2\gamma e^{(\gamma-\beta)u/2}}{(\gamma-\beta)(e^{\gamma u}-1)+2\gamma}\right), \\ B(u) &= \frac{2(e^{\gamma u}-1)}{(\gamma-\beta)(e^{\gamma u}-1)+2\gamma}, \\ \gamma &= \sqrt{\beta^2 + 2\sigma^2}. \end{aligned}$$

3. Consider again the reduced-form credit-risk models introduced in the lecture. The aim of the exercise is to compute the price of a defautable zero-coupon bond with maturity T. Let $\delta \in (0,1)$, let τ denote the default time and consider the following three cases:

- Zero recovery: the cash flow at T is $\mathbb{I}_{\{\tau > T\}}$
- Partial recovery at maturity: the cash flow at T is $\mathbb{I}_{\{\tau > T\}} + \delta \mathbb{I}_{\{\tau \leq T\}}$
- Partial recovery at default: the cash flow is

$$\begin{cases} 1 & \text{at} \quad T, \quad \text{if} \quad \tau > T, \\ \delta & \text{at} \quad \tau, \quad \text{if} \quad \tau \leq T. \end{cases}$$

[1] That is, if we condition on a particular realization $\lambda(\cdot, \omega)$ of the intensity, the jump process becomes an inhomogeneous Poisson process with intensity $\lambda(s, \omega)$.

Siehe nächstes Blatt!

In this exercise we choose the short rate process r to be a CIR process, i.e.,

$$dr(t) = (b + \beta r(t))dt + \sigma\sqrt{r(t)}dW(t), \quad r(0) \geq 0,$$

where $b \geq 0, \beta \in \mathbb{R}, \sigma > 0$ such that $2b \geq \sigma^2$ and W is a $(\mathbb{Q}, \mathcal{F})$ Brownian motion. Furthermore, we choose the intensity process λ to be affine, i.e.,

$$\lambda(t) = c_0 + c_1 r(t),$$

where $c_0, c_1 \geq 0$ are two positive constants.

a) Show that the price of a corporate zero-coupon bond with zero recovery is given by

$$C^0(T) = e^{-A(T)-B(T)r(0)},$$

where

$$\begin{aligned} A(u) &= c_0 u - \frac{2b(1+c_1)}{\sigma^2}\log\left(\frac{2\gamma e^{(\gamma-\beta)u/2}}{(\gamma-\beta)(e^{\gamma u}-1)+2\gamma}\right), \\ B(u) &= \frac{2(e^{\gamma u}-1)}{(\gamma-\beta)(e^{\gamma u}-1)+2\gamma}(1+c_1), \\ \gamma &= \sqrt{\beta^2 + 2(1+c_1)\sigma^2}. \end{aligned}$$

b) Show that the price of a corporate zero-coupon bond with partial recovery at maturity is

$$C(T) = (1-\delta)C^0(T) + \delta P(0,T),$$

where $P(0,T)$ denotes the price of a default free zero-coupon bond at time 0.

c) Show that the price of a corporate zero-coupon bond with partial recovery at default is given by

$$C^D(T) = C^0(T) + \delta\Pi(T),$$

with

$$\Pi(T) = \left(\frac{c_0}{1+c_1}\int_0^T e^{-A(u)-B(u)r(0)}du + \frac{c_1}{1+c_1}\left(1 - e^{-A(T)-B(T)r(0)}\right)\right).$$

4. Matlab Exercise We consider the homogeneous Bernoulli mixture model with factor X, i.e., Y_1 and Y_2 are independent Bernoulli random variables with common success factor $p(X)$. For $i \in \{1,2\}$ we assume that the firm value is given by

$$V_i(t) = V_i(0)e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i B_i(t)},$$

where $\mu_i, \sigma_i > 0$ are real constants and (B_1, B_2) denotes a two-dimensional Brownian motion with constant correlation $\varrho > 0$. Hence, in the corresponding Merton model the default probabilities are given by

$$p_i = \mathbb{P}\left[W_i(1) < C_i\right] = \Phi(C_i),$$

where $W_i(1) = \frac{\log\left(V_i(1)/V_i(0) - (\mu_i - \sigma_i^2/2)\right.}{\sigma_i}$ is the standardized asset return and $C_i = \frac{\log L_i - (\mu_i - \sigma_i^2/2)}{\sigma_i}$ the standardized face value of the debt D_i and $L_i = D_i/V_i(0)$ is the initial leverage ratio. Therefore, there exists independent standard normal random variables X, Z_1 and Z_2 such that

$$W_i(1) = \sqrt{\varrho}X + \sqrt{1-\varrho}Z_i.$$

Moreover, we assume that Y_1 and Y_2 are exchangeable, e.g., $p_1 = p_2 = p$.

a) Compute the joint default probability $\mathbb{P}\left[Y_1 = 1, Y_2 = 1\right]$.

b) For simplicity, we assume that the loss given default of each company is 1. Compute the distribution of $L_2 = Y_1 + Y_2$.

c) For the following set of parameters,

$$\mu_1 = \mu_2 = 0.01, \sigma_1 = \sigma_2 = 0.2, \varrho = 0.5, V_0^1 = V_0^2 = 100, D^1 = D^2 = 90$$

plot the empirical distribution function of L_2 using simulations ($N = 10^4$) and your result from part b) using numerical integration.

Happy Holidays!