ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Exercise Sheet 2

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ carrying a (standard) Brownian motion $W = (W_t)_{t\geq 0}$.

1. Let $n \in \mathbb{N}$ be arbitrary and let $(t_i)_{i\in\{0,\dots,n\}}$ be an $(n+1)$-tuple of positive real numbers with the property that $0 = t_0 < t_1 < \dots < t_n < \infty$. Consider a simple integrand

$$h : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}, (\omega, t) \longmapsto h(\omega, t)$$

of the form[1]

$$h(\omega, t) = \sum_{i=1}^{n} \varphi_i(\omega)\mathbb{I}_{(t_{i-1},t_i]}(t), \tag{1}$$

where φ_i are $\mathcal{F}_{t_{i-1}}$-measurable random variables, bounded in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and let

$$(h \bullet W)_t := \int_0^t h(s)dW_s = \sum_{i=1}^{n} \varphi_i(W_{t_i\wedge t} - W_{t_{i-1}\wedge t})$$

denote the stochastic integral of h with respect to Brownian motion W.

a) Show that the process $h \bullet W$ is a continuous martingale.

b) Show the Itô isometry for $h \bullet W$, i.e.

$$\mathbb{E}\left[\left(\int_0^\infty h(s)dW_s\right)^2\right] = \mathbb{E}\left[\int_0^\infty |h(s)|^2\, ds\right]. \tag{2}$$

2. (Yor's Formula) Let X be an Itô process and let $\mathcal{E}(X)$ denote its stochastic exponential

$$\mathcal{E}_t(X) := e^{X(t)-X(0)-\frac{1}{2}\langle X,X\rangle_t}.$$

Show the identity

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y)e^{\langle X,Y\rangle}.$$

[1] We assume that $h_0(\omega) = \varphi_1(\omega)$.

Bitte wenden!

3. Consider the *Ornstein-Uhlenbeck process*

$$X_t = xe^{-\lambda t} + \nu(1 - e^{-\lambda t}) + \int_0^t \sigma e^{\lambda(s-t)} dW_s, \quad t \geq 0 \tag{3}$$

for an $x \in \mathbb{R}$, where the parameters ν and $\lambda, \sigma > 0$ take real values.

a) Show that X satisfies the Ornstein-Uhlenbeck stochastic differential equation:

$$dX_t = \lambda(\nu - X_t)dt + \sigma dW_t, \quad X_0 = x.$$

b) Calculate the mean and variance functions of X:

$$T \mapsto \mathbb{E}[X_T], \quad \text{and} \quad T \mapsto \text{Var}[X_T].$$

c) For $\tau > 0$ consider the rescaled $AR(1)$ process $(Y^\tau)_{n \geq 0}$ defined by

$$Y_n^\tau = c_\tau + \varphi_\tau Y_{n-1}^\tau + \sigma_\tau \varepsilon_n, \quad Y_0 = x$$

with $c_\tau = \lambda \nu \tau, \varphi_\tau = 1 - \lambda \tau$, $\sigma_\tau = \sigma\sqrt{\tau}$ and ε_n are i.i.d standard Gaussian random variables. Verify that the corresponding mean and variance of $Y_{[t/\tau]}^\tau$ indeed converge to its Ornstein-Uhlenbeck counterpart, i.e,

$$\mathbb{E}[Y_{[t/\tau]}^\tau] \to \mathbb{E}[X_t] \quad \text{and} \quad Var[Y_{[t/\tau]}^\tau] \to Var[X_t] \quad \text{as} \quad \tau \to 0,$$

where $[x]$ denotes the integer part of x.
Note: It can be shown that the rescaled process $Y_{[t/\tau]}^\tau$ converges weakly to the Ornstein-Uhlenbeck process X_t as $\tau \to 0$.

4. Consider again the Ornstein-Uhlenbeck process X in the setting of Ex 2-3.

a) Show that for any $T > 0$ the distribution of X_T is given by

$$X_T \sim \mathcal{N}\left(xe^{-\lambda T} + \nu(1 - e^{-\lambda T}), \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda T})\right)$$

by proceeding as follows:

- Show in general for arbitrary $T > 0$, that if

$$f : [0, T] \longrightarrow \mathbb{R} \in C^0([0, T]),$$

$$\text{then} \quad \int_0^T f(s) dW_s \sim \mathcal{N}\left(0, \int_0^T (f(s))^2 ds\right). \tag{4}$$

Siehe nächstes Blatt!

- Conclude the statement using the first step and Ex 2-3 b).

Hint: For the first point approximate the process by simple functions, then use Lévy's continuity theorem and the fact that the characteristic function of a random variable uniquely characterizes its distribution.

b) Compute $\mathbb{E}[X_T^+]$ explicitly

5. Matlab Implementation Given a finite time horizon $T = 1$, the aim of this exercise is to simulate the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process on the time interval $[0, T]$ using the *Euler-Maruyama scheme*. We define an equidistant decomposition $\{0 = t_0 < \ldots < t_n = T\}$ of the interval $[0, T]$ by setting

$$t_i := \frac{i}{M} T, \quad i = 0, \ldots, M = 10^3.$$

If X is a process on the interval $[0, T]$ satisfying the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

with initial condition $X_0 = x$ for an $x \in \mathbb{R}$, and $t_0 = 0 < t_1 < \ldots < t_M = T$ is a given discretization of the time interval $[0, T]$, then an *Euler-Maruyama approximation*[2] of X is given by the iterative scheme: $X_0 = x$ and

$$X_{t_{i+1}} = X_{t_i} + a(t_i, X_{t_i})(t_{i+1} - t_i) + b(t_i, X_{t_i})(W_{t_{i+1}} - W_{t_i}), \quad i = 0, \ldots, M - 1.$$

a) Simulate 10 sample paths of the OU-process X from Ex 2-3 with $\lambda = 1$, $\nu = 1.2$, $\sigma = 0.3$ and $X_0 = 1$.

b) Use Monte-Carlo simulation ($N = 10^5$) to compute $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^+]$

c) Consider the *Cox-Ingersoll-Ross* process Y defined by the following SDE:

$$dY_t = \lambda(\nu - Y_t)dt + \sigma\sqrt{Y_t}dW_t, \quad Y_0 = y.$$

Assuming $2\lambda\nu \geq \sigma^2$ repeat the tasks (a) and (b) for the CIR process. Is there a potential problem for the simulation procedure?

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[2] As a reference for the Euler-Maruyama approximation see for example Section 3.2 of *Numerical Solution of SDE Through Computer Experiments* (Kloeden, Platen, Schurz).