

Interest Rate Theory Solution Sheet 2

1. Since adapted processes with left-continuous paths are predictable (in particular progressively measurable)¹, we are in the situation of Theorem 4.1 (Filipović: Term-Structure Models) presented in the lecture. First of all, we bring the stochastic integral to a form which is more convenient for calculations: There exists a $k \leq n$ such that $t_{k-1} < t \leq t_k$ and thus the stochastic integral can be rewritten as

$$\begin{aligned} (h \bullet W)_t &= \sum_{i=1}^n \varphi_i(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}) \\ &= \sum_{i=1}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}). \end{aligned}$$

- a) Recognizing that $h \bullet W$ is a linear combination of compositions of continuous functions, \mathbb{P} -a.s. continuity of paths is immediate. In order to show that $h \bullet W$ is a continuous-time martingale, we need to verify the following properties:

1. $(h \bullet W)_t$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$.
2. For all $t \geq 0$ the stochastic integral $(h \bullet W)_t$ is integrable, i.e.

$$\mathbb{E}[|(h \bullet W)_t|] < \infty.$$

3. For all $0 \leq s < t < \infty$

$$\mathbb{E}[(h \bullet W)_t | \mathcal{F}_s] = (h \bullet W)_s, \quad \mathbb{P} - a.s.$$

Property 1. is immediate, it holds since Brownian motion is adapted to the given filtration and by construction of h , where the φ_i , are in particular $\mathcal{F}_{t_{i-1}}$ -measurable.

¹See for example: Brownian Motion and Stochastic Calculus script (2012) p.4.

Bitte wenden!

To show 2., consider

$$\begin{aligned}
\mathbb{E}[|(h \bullet W)_t|] &= \mathbb{E}\left[\left|\sum_{i=1}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}})\right|\right] \\
&\leq \sum_{i=1}^{k-1} \mathbb{E}[|\varphi_i(W_{t_i} - W_{t_{i-1}})|] + \mathbb{E}[|\varphi_k(W_t - W_{t_{k-1}})|] \\
&\leq \sum_{i=1}^{k-1} \left(\mathbb{E}[|\varphi_i|^2] \mathbb{E}[|W_{t_i} - W_{t_{i-1}}|^2]\right)^{1/2} + \left(\mathbb{E}[|\varphi_k|^2] \mathbb{E}[|W_t - W_{t_{k-1}}|^2]\right)^{1/2} \\
&\leq \sum_{i=1}^{k-1} \left(\mathbb{E}[|\varphi_i|^2] \mathbb{E}[|W_{t_i} - W_{t_{i-1}}|^2]\right)^{1/2} + \left(\mathbb{E}[|\varphi_k|^2] \mathbb{E}[|W_t - W_{t_{k-1}}|^2]\right)^{1/2} \\
&\leq \sum_{i=1}^{k-1} \left(\mathbb{E}[|\varphi_i|^2] (t_i - t_{i-1})\right)^{1/2} + \left(\mathbb{E}[|\varphi_k|^2] (t - t_{k-1})\right)^{1/2} < \infty.
\end{aligned}$$

For 3. if $t_{k-1} \leq s < t \leq t_k$ the calculation is straightforward

$$\begin{aligned}
\mathbb{E}[(h \bullet W)_t | \mathcal{F}_s] &= \mathbb{E}\left[\sum_{i=1}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_s\right] \\
&= \sum_{i=1}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \mathbb{E}[\varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_s] \\
&= \sum_{i=1}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_s - W_{t_{k-1}}) \\
&= (h \bullet W)_s.
\end{aligned}$$

In case $t_{l-1} \leq s < t_l < t$ for an $l \leq k-1$, we similarly have

$$\begin{aligned}
\mathbb{E}[(h \bullet W)_t | \mathcal{F}_s] &= \mathbb{E}\left[\sum_{i=1}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_s\right] \\
&= \sum_{i=1}^{l-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \mathbb{E}\left[\sum_{i=l}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_s\right],
\end{aligned} \tag{1}$$

Siehe nächstes Blatt!

where

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i=l}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_s \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\sum_{i=l}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_{t_l} \right] | \mathcal{F}_s \right] \\
&= \mathbb{E} \left[\varphi_l(W_{t_l} - W_{t_{l-1}}) + \varphi_{l+1} \underbrace{\mathbb{E} [W_{t_{l+1}} | \mathcal{F}_{t_l}]}_{=W_{t_l}} - \varphi_{l+1}W_{t_l} + \right. \\
&\quad \left. + \mathbb{E} \left[\sum_{i=l+2}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_{t_l} \right] | \mathcal{F}_s \right] \\
&= \mathbb{E} \left[\varphi_l(W_{t_l} - W_{t_{l-1}}) + 0 + \varphi_{l+2} \underbrace{\mathbb{E} [W_{t_{l+2}} | \mathcal{F}_{t_{l+1}}]}_{=W_{t_{l+1}}} - \varphi_{l+2}W_{t_{l+1}} + \right. \\
&\quad \left. + \mathbb{E} \left[\mathbb{E} \left[\sum_{i=l+3}^{k-1} \varphi_i(W_{t_i} - W_{t_{i-1}}) + \varphi_k(W_t - W_{t_{k-1}}) | \mathcal{F}_{t_{l+1}} \right] | \mathcal{F}_{t_l} \right] | \mathcal{F}_s \right] \\
&= \dots \\
&= \mathbb{E} [\varphi_l(W_{t_l} - W_{t_{l-1}}) | \mathcal{F}_s] = \varphi_l \mathbb{E} [W_{t_l} | \mathcal{F}_s] - \varphi_l W_{t_{l-1}} = \varphi_l(W_s - W_{t_{l-1}}).
\end{aligned}$$

Using this for the last term in (1) yields

$$\mathbb{E} [(h \bullet W)_t | \mathcal{F}_s] = (h \bullet W)_s$$

for the integral.

b)

$$\begin{aligned}
\mathbb{E} \left[\left(\int_0^\infty h(s) dW_s \right)^2 \right] &= \mathbb{E} \left[\left(\sum_{i=1}^n \varphi_i(W_{t_i} - W_{t_{i-1}}) \right)^2 \right] \\
&= \sum_{i=1}^n \sum_{j=1}^n \mathbb{E} [\varphi_i \varphi_j (W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}})],
\end{aligned} \tag{2}$$

where the summands with $j < i$ (and $i < j$) vanish, since

$$\begin{aligned}
& \mathbb{E} [\varphi_i \varphi_j (W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}})] \\
&= \mathbb{E} [\mathbb{E} [\varphi_i \varphi_j (W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}}) | \mathcal{F}_{t_{i-1}}]] \\
&= \mathbb{E} [\varphi_i \varphi_j (W_{t_j} - W_{t_{j-1}}) (\underbrace{\mathbb{E} [W_{t_i} | \mathcal{F}_{t_{i-1}}]}_{=W_{t_{i-1}}}) - W_{t_{i-1}}] = 0
\end{aligned}$$

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and for $i = j$

$$\begin{aligned}
\mathbb{E} [\varphi_i^2 (W_{t_i} - W_{t_{i-1}})^2] &= \mathbb{E} [\mathbb{E} [\varphi_i^2 (W_{t_i} - W_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&= \mathbb{E} [\varphi_i^2 \mathbb{E} [(W_{t_i} - W_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&= \mathbb{E} [\varphi_i^2 \mathbb{E} [(W_{t_i} - W_{t_{i-1}})^2]] \\
&= \mathbb{E} [\varphi_i^2 \mathbb{E} [(W_{t_i - t_{i-1}})^2]] \\
&= \mathbb{E} [\varphi_i^2] (t_i - t_{i-1}),
\end{aligned}$$

therefore (2) simplifies to

$$\sum_{i=1}^n \mathbb{E} [\varphi_i^2] (t_i - t_{i-1}) = \mathbb{E} \left[\sum_{i=1}^n |\varphi_i|^2 (t_i - t_{i-1}) \right] = \mathbb{E} \left[\int_0^\infty |h(s)|^2 ds \right].$$

2. a) To ease notation let's assume $X(0) = 0$. Notice that if we know the form of the stochastic exponential, i.e. that

$$\mathcal{E}_t(X) = e^{X(t) - \frac{1}{2} \langle X, X \rangle_t},$$

then write out the bracket process using bilinearity,

$$\langle X + Y, X + Y \rangle = \langle X, X \rangle + 2\langle X, Y \rangle + \langle Y, Y \rangle$$

to find formally

$$\begin{aligned}
\mathcal{E}_t(X) \mathcal{E}_t(Y) &= e^{X(t) - \frac{1}{2} \langle X, X \rangle_t} e^{Y(t) - \frac{1}{2} \langle Y, Y \rangle_t} \\
&= e^{X(t) + Y(t) - \frac{1}{2} \langle X, X \rangle_t - \frac{1}{2} \langle Y, Y \rangle_t - \langle X, Y \rangle_t + \langle X, Y \rangle_t} \\
&= e^{(X+Y)(t) - \frac{1}{2} \langle X+Y, X+Y \rangle_t + \langle X, Y \rangle_t} \\
&= \mathcal{E}_t(X + Y) e^{\langle X, Y \rangle_t}.
\end{aligned}$$

If we characterize the stochastic exponential $\mathcal{E}_t(Z)$ of an Itô process Z by its property that it is the unique solution of the stochastic differential equation

$$dU(t) = U(t) dZ(t). \tag{3}$$

Then we can show the statement by proceeding as follows:

$$\mathcal{E}(X) \mathcal{E}(Y) \stackrel{i)}{=} \mathcal{E}(X + Y + \langle X, Y \rangle) \stackrel{ii)}{=} \mathcal{E}(X + Y) e^{\langle X, Y \rangle}.$$

Siehe nächstes Blatt!

i) By the product rule

$$\begin{aligned}
\mathcal{E}_t(X)\mathcal{E}_t(Y) &= \mathcal{E}_0(X)\mathcal{E}_0(Y) + \int_0^t \mathcal{E}_s(X)\mathcal{E}_s(Y)dY_s + \\
&\quad + \int_0^t \mathcal{E}_s(Y)\mathcal{E}_s(X)dX_s + \langle \mathcal{E}(X), \mathcal{E}(Y) \rangle_t \\
&= \mathcal{E}_0(X)\mathcal{E}_0(Y) + \int_0^t \mathcal{E}_s(X)\mathcal{E}_s(Y)d(X+Y)_s \\
&\quad + \mathcal{E}_t(X)\mathcal{E}_t(Y)\langle X, Y \rangle_t.
\end{aligned}$$

Therefore, $U(t) = (\mathcal{E}(X)\mathcal{E}(Y))_t$ satisfies (3) with $Z = X + Y + \langle X, Y \rangle$.

ii) On the other hand, since $e^{\langle X, Y \rangle}$ is of finite variation, the product rule applies with vanishing quadratic variation term for

$$\begin{aligned}
d\left(\mathcal{E}(X+Y)e^{\langle X, Y \rangle}\right)_t &= \mathcal{E}(X+Y)_t d e^{\langle X, Y \rangle}_t + e^{\langle X, Y \rangle}_t d\mathcal{E}(X+Y)_t \\
&= \mathcal{E}(X+Y)_t e^{\langle X, Y \rangle}_t d\langle X, Y \rangle_t + e^{\langle X, Y \rangle}_t \mathcal{E}(X+Y)_t d(X+Y)_t \\
&= \mathcal{E}(X+Y)_t e^{\langle X, Y \rangle}_t d(X+Y + \langle X, Y \rangle)_t.
\end{aligned}$$

This proves that $U(t) = (\mathcal{E}(X+Y)e^{\langle X, Y \rangle})_t$ also solves (3) with $Z = X + Y + \langle X, Y \rangle$. By uniqueness therefore

$$(\mathcal{E}(X)\mathcal{E}(Y))_t = (\mathcal{E}(X+Y)e^{\langle X, Y \rangle})_t$$

as claimed.

3. a) Consider the function $f(x, t) = xe^{\lambda t}$. Itô's formula applied to f yields

$$\begin{aligned}
f(X_t, t) &= X_0 e^{\lambda 0} + \int_0^t X_s \lambda e^{\lambda s} ds + \int_0^t e^{\lambda s} dX_s \\
&= X_0 e^{\lambda 0} + \int_0^t X_s \lambda e^{\lambda s} ds + \int_0^t e^{\lambda s} \lambda (\nu - X_s) ds + \int_0^t e^{\lambda s} \sigma dW_s \\
&= X_0 e^{\lambda 0} + \int_0^t e^{\lambda s} \lambda \nu ds + \int_0^t e^{\lambda s} \sigma dW_s \\
&= X_0 e^{\lambda 0} + \nu(e^{\lambda t} - 1) + \int_0^t e^{\lambda s} \sigma dW_s.
\end{aligned}$$

Now multiplying both sides by $e^{-\lambda t}$ and inserting the initial value $X_0 = x$ \mathbb{P} -a.s. gives

$$f(X_t, t)e^{-\lambda t} = X_t = xe^{-\lambda t} + \nu(1 - e^{-\lambda t}) + \int_0^t e^{\lambda(s-t)} \sigma dW_s, \quad t \geq 0.$$

Bitte wenden!

- b) To compute $\mathbb{E}[X_t]$ we first show that $(\int_0^t \sigma e^{\lambda(s-t)} dW_s)_{0 \leq t \leq T}$ is a $(\mathbb{P}, \mathcal{F})$ -martingale on $[0, T]$ with mean 0. We notice that the integrand is predictable (it is continuous and adapted) and locally bounded. Therefore, the stochastic integral is a local-martingale. Since

$$\mathbb{E} \left[\int_0^T \sigma^2 e^{2\lambda(s-T)} ds \right] = \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda T}) < \infty,$$

it is even a true martingale by Theorem 4.1/d). Thus,

$$\mathbb{E}[X_T] = xe^{-\lambda T} + \nu(1 - e^{-\lambda T}).$$

Moreover, using Itô's isometry we have

$$\begin{aligned} \text{Var}[X_T] &= \mathbb{E}[(X_T - \mathbb{E}[X_T])^2] \\ &= \mathbb{E} \left[\left(\sigma \int_0^T e^{\lambda(s-T)} dW_s \right)^2 \right] \\ &= \sigma^2 \mathbb{E} \left[\int_0^T e^{2\lambda(s-T)} ds \right] \\ &= \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda T}). \end{aligned}$$

- c) Applying the definition of Y^τ recursively n times we have

$$Y_n^\tau = c_\tau \sum_{k=0}^{n-1} \varphi_\tau^k + \varphi_\tau^n Y_0 + \sigma_\tau \sum_{k=0}^{n-1} \varphi_\tau^k \varepsilon_{n-k}.$$

Inserting the values for $c_\tau, \varphi_\tau, \sigma_\tau$ and setting $n = [t/\tau]$ yields

$$Y_{[t/\tau]}^\tau = \nu\lambda\tau \sum_{k=0}^{[t/\tau]-1} (1 - \lambda\tau)^k + (1 - \lambda\tau)^{[t/\tau]} x + \sigma\sqrt{\tau} \sum_{k=0}^{[t/\tau]-1} (1 - \lambda\tau)^k \varepsilon_{[t/\tau]-k}.$$

Hence,

$$\begin{aligned} \mathbb{E}[Y_{[t/\tau]}^\tau] &= \nu\lambda\tau \frac{1 - (1 - \lambda\tau)^{[t/\tau]}}{1 - (1 - \lambda\tau)} + x(1 - \lambda\tau)^{[t/\tau]} \\ &\rightarrow \nu(1 - e^{-\lambda t}) + xe^{-\lambda t} \quad \text{as } \tau \rightarrow 0 \\ \text{Var}[Y_{[t/\tau]}^\tau] &= \sigma^2\tau \sum_{k=0}^{[t/\tau]-1} (1 - \lambda\tau)^{2k} \\ &= \sigma^2\tau \frac{1 - (1 - \lambda\tau)^{2[t/\tau]}}{1 - (1 - \lambda\tau)^2} \\ &= \sigma^2 \frac{1 - (1 - \lambda\tau)^{2[t/\tau]}}{2\lambda - \lambda^2\tau} \\ &\rightarrow \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t}) \quad \text{as } \tau \rightarrow 0. \end{aligned}$$

Siehe nächstes Blatt!

4. a) Consider the partition $0 = t_0 < \dots < t_i = i \frac{T}{n} < \dots < t_n = T$ on the given time horizon $[0, T]$ and the sequence

$$\tilde{h}_n(s) := f(s) - \sum_{i=1}^n f(t_{i-1}) \mathbb{I}_{(t_{i-1}, t_i]}(s) \quad s \in [0, T].$$

Setting $h_n(s) = \tilde{h}_n(s) \mathbb{I}_{[0, T]}(s)$, we extend the domain of definition to $s \in [0, \infty)$. Then Dominated Convergence Theorem (cf. Theorem 4.1/e) of Filipović: Term-Structure Models) is applicable: by continuity of f the boundedness requirements are satisfied and also

$$\lim_{n \rightarrow \infty} h_n(s) = 0$$

pointwise. It follows that

$$\int_0^T f(s) dW_s = \lim_{n \rightarrow \infty} \underbrace{\sum_{i=1}^n f\left((i-1)\frac{T}{n}\right) (W_{i\frac{T}{n}} - W_{(i-1)\frac{T}{n}})}_{Z_n} \quad \text{in probability.}$$

For any $n \geq 0$, the distribution is

$$Z_n \sim \mathcal{N}(\nu_n, \sigma_n^2)$$

with $\nu_n = 0$ and with $\sigma_n^2 = \sum_{i=1}^n \left(f\left((i-1)\frac{T}{n}\right)\right)^2 \frac{T}{n}$ by independence of Brownian motion increments. The characteristic functions of the normal random variables Z_n are therefore given by

$$\varphi_{Z_n}(t) = \mathbb{E}[e^{itZ_n}] = e^{it\nu_n - \frac{1}{2}\sigma_n^2 t^2} = e^{\frac{1}{2}\sigma_n^2 t^2}.$$

It holds furthermore that

$$\sigma^2 := \lim_{n \rightarrow \infty} \sigma_n^2 = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(f\left((i-1)\frac{T}{n}\right)\right)^2 \frac{T}{n} = \int_0^T (f(s))^2 ds.$$

Since the sequence of characteristic functions $\varphi_{Z_n}(t) = e^{-\frac{1}{2}\sigma_n^2 t^2}$ converges pointwise to the continuous function $e^{-\frac{1}{2}\sigma^2 t^2}$, we can conclude by *Lévy's continuity theorem* that the sequence Z_n converges in distribution to a random variable Z with characteristic function $\varphi_Z(t) = e^{-\frac{1}{2}\sigma^2 t^2}$. Hereby we know that the limit $Z = \int_0^T f(s) dW_s$ is normally distributed with mean zero and variance $\int_0^T (f(s))^2 ds$, and the assertion

$$\int_0^T f(s) dW_s \sim \mathcal{N}\left(0, \int_0^T (f(s))^2 ds\right) \quad (4)$$

is proven. The result now follows immediately from Ex 2-3 b).

Bitte wenden!

b) Let $\tilde{\mu}_T$ and $\tilde{\sigma}_T^2$ denote the expectation and variance of X_T from Ex 2-3 b), then

$$\begin{aligned}
 \mathbb{E}[X_T^+] &= \int_0^\infty \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} y e^{-\frac{(y-\tilde{\mu})^2}{2\tilde{\sigma}^2}} dy \\
 &= \int_{-\frac{\tilde{\mu}}{\tilde{\sigma}}}^\infty \frac{1}{\sqrt{2\pi}} (\tilde{\mu} + \tilde{\sigma}z) e^{-\frac{z^2}{2}} dz \\
 &= \tilde{\mu} \left(1 - \Phi \left(-\frac{\tilde{\mu}}{\tilde{\sigma}} \right) \right) + \frac{\tilde{\sigma}}{\sqrt{2\pi}} e^{-\frac{\tilde{\mu}^2}{2\tilde{\sigma}^2}} \\
 &= \tilde{\mu} \Phi \left(\frac{\tilde{\mu}}{\tilde{\sigma}} \right) + \frac{\tilde{\sigma}}{\sqrt{2\pi}} e^{-\frac{\tilde{\mu}^2}{2\tilde{\sigma}^2}}
 \end{aligned}$$

where $\Phi(x)$ denotes the cumulative distribution function of a standard normal random variable.

5. see OUprocess.m and CIR1.m

6. Matlab Files

```

1 function OUprocess
2 % In this exercise we simulated N paths of a OU process
3 % dX_t = \lambda (\nu - X_t) dt + \sigma dW_t, X_0 = x and
   simulate the
4 % expectation of X_1, X_1^2, X_1^+
5 tic
6 % parameter input
7 % horizon
8 T=1;
9 % sample size
10 Nsimu=10^5;
11 Nplot=Nsimu;
12 % grid points
13 M=10^3;
14 % volatility
15 sigma=0.3;
16 lambda=1;
17 nu=1.2;
18 x=1;
19 % time step
20 dt= T/M;
21
22 % theoretical value for the expectation, second moment
   and pos part

```

Siehe nächstes Blatt!


```

23 mutilde = x*exp(-lambda*T)+ nu*(1-exp(-lambda*T));
24 sigmatilde= sqrt(sigma^2/(2*lambda)*(1-exp(-2*lambda*T))
    );
25 theoreticalvalueexp= mutilde;
26 theoreticalvaluesec= sigmatilde^2+mutilde^2;
27 theoreticalvaluepos= mutilde*normcdf(mutilde/sigmatilde)
    +sigmatilde/sqrt(2*pi)*exp(-1/2*(mutilde/sigmatilde)
    ^2);
28 %% Simulation
29 % BM
30 BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
31 OU = [x*ones(1,Nplot);zeros(M,Nplot)];
32 % the process X^(b)
33 for i =1:M
34     OU(i+1,:)=OU(i,:)+lambda*(nu-OU(i,:))*dt+ sigma.*(BM
        (i+1,:)-BM(i,:));
35 end
36
37 %plot the first 10 sample paths
38 timegrid= 0:dt:T;
39 plot(timegrid,OU(:,1:10))
40
41 %compute simulated value
42 simulatedvalueexp= mean(OU(end,:));
43 simulatedvaluesec= mean(OU(end,:).^2);
44 simulatedvaluepos= mean(subplus(OU(end,:)));
45
46 disp('Exact values: Expectation/2.Moment/pos. part')
47 disp([theoreticalvalueexp;theoreticalvaluesec;
    theoreticalvaluepos])
48 disp('Estimated value: Expectation/2.Moment/pos. part')
49 disp([simulatedvalueexp;simulatedvaluesec;
    simulatedvaluepos])
50
51 %estimated variance
52 %estvarexp= var(OU(end,:));
53 %estvarsec= var(OU(end,:).^2);
54 %estvarpos= var(subplus(OU(end,:)));
55 % confidence interval using CLT
56 %cfplusexp=simulatedvalueexp+1.96*sqrt(estvarexp/Nsimu);
57 %cfminusexp=simulatedvalueexp-1.96*sqrt(estvarexp/Nsimu)
    ;

```

Bitte wenden!

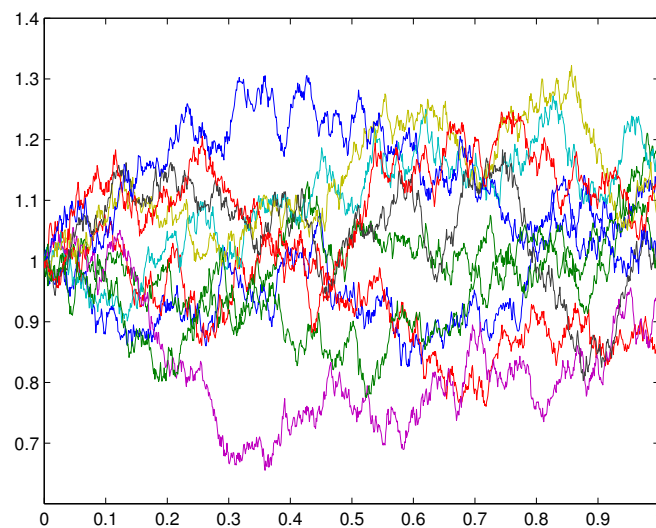


Abbildung 1: 10 sample paths of the OU process

```

58 %cfplussec=simulatedvaluesec+1.96*sqrt(estvarsec/Nsimu);
59 %cfminussec=simulatedvaluesec-1.96*sqrt(estvarsec/Nsimu)
    ;
60 %cfpluspos=simulatedvaluepos+1.96*sqrt(estvarpos/Nsimu);
61 %cfminuspos=simulatedvaluepos-1.96*sqrt(estvarpos/Nsimu)
    ;
62
63 %disp('Confidence interval: ')
64 %disp([cfminusexp, cfplusexp; cfminussec, cfplussec;
        cfminuspos, cfpluspos])
65
66 toc

```

7. Matlab Files

```

1 function CIR1
2 % In this exercise we simulated N paths of a CIR process
3 %  $dX_t = \lambda (\nu - X_t) dt + \sigma \sqrt{X_t} dW_t$ ,
    $X_0 = x$  and simulate the
4 % expectation of  $X_1$ ,  $X_1^2$ ,  $X_1^4$ 
5 tic
6 %% parameter input
7 % horizon

```

Siehe nächstes Blatt!

```

8 T=1;
9 % sample size
10 Nsimu=10^5;
11 Nplot=Nsimu;
12 % grid points
13 M=10^3;
14 % volatility
15 lambda=1;
16 nu=1.2;
17 sigma=0.3;
18 x=1;
19 % time step
20 dt= T/M;
21
22 % Check the Feller condition
23 check = 2*lambda*nu >= sigma^2;
24 if check == 1
25     disp('Feller condition is assumed')
26 elseif check==0
27     disp('Feller condition is not satisfied')
28 end
29
30
31 % theoretical value for the expectation , second moment
    and pos part
32 constc = 4*lambda/(sigma^2*(1-exp(-lambda*T)));
33 constv = 4*lambda*nu/(sigma^2);
34 constlambda= constc*x*exp(-lambda*T);
35 % constc_T*X_T is non central chi square distributed (
    constv , constlambda)
36 % E(non-central chisquare distribution) = constv+
    constlambda;
37 theoreticalvalueexp= (constv+constlambda)/constc;
38 % or equivalently ,
39 %theoreticalvalueexp= x*exp(-lambda*T)+nu*(1-exp(-lambda
    *T));
40 % Var(non-central chisquare distribution) = 2(constv+2*
    constlambda)
41 theoreticalvaluesec= (2*(constv+2*constlambda)+(constv+
    constlambda)^2)/(constc^2);
42 % or equivalently
43 %theoreticalvaluesec= x*sigma^2/lambda*(exp(-lambda*T)-

```

Bitte wenden!

```

        exp(-2*lambda*T))+nu*sigma^2/(2*lambda)*(1-exp(-lambda
        *T))^2+theoreticalvalueexp^2;
44 % Use numerical integration to obtain a theoretical
    value
45 % Since the Feller condition is assumed, E[X^+_1] must
    be equal to E[X_1]
46 theoreticalvaluepos=integral(@(x) x.*ncx2pdf(x,constv ,
    constlambda),0,Inf)/constc;
47 %% Simulation
48 % BM
49 BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
50 CIR = [x*ones(1,Nplot);zeros(M,Nplot)];
51 % the process X^(b)
52 for i =1:M
53     CIR(i+1,:)=CIR(i,:)+lambda*(nu-CIR(i,:))*dt+ sigma.*
        sqrt(CIR(i,:)).*(BM(i+1,:)-BM(i,:));
54 end
55
56 %plot the first 10 sample paths
57 timegrid= 0:dt:T;
58 plot(timegrid,CIR(:,1:10))
59
60 %compute simulated value
61 simulatedvalueexp= mean(CIR(end,:));
62 simulatedvaluesec= mean(CIR(end,:).^2);
63 simulatedvaluepos= mean(subplus(CIR(end,:)));
64
65 disp('Exact values: Expectation/2.Moment/pos. part')
66 disp([theoreticalvalueexp;theoreticalvaluesec;
    theoreticalvaluepos])
67 disp('Estimated value: Expectation/2.Moment/pos. part')
68 disp([simulatedvalueexp;simulatedvaluesec;
    simulatedvaluepos])
69
70 %estimated variance
71 %estvarexp= var(CIR(end,:));
72 %estvarsec= var(CIR(end,:).^2);
73 %estvarpos= var(subplus(CIR(end,:)));
74 % confidence interval using CLT
75 %cfplusexp=simulatedvalueexp+1.96*sqrt(estvarexp/Nsimu);
76 %cfminusexp=simulatedvalueexp-1.96*sqrt(estvarexp/Nsimu)
    ;

```

Siehe nächstes Blatt!

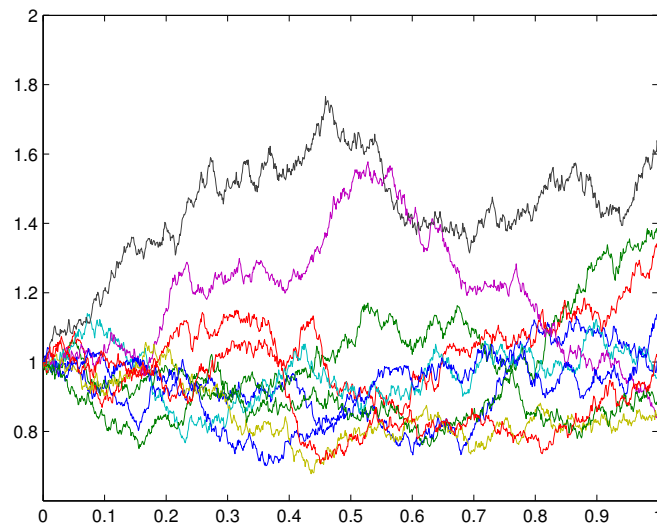


Abbildung 2: 10 sample paths of the CIR process

```

77 %cfplussec=simulatedvaluesec+1.96*sqrt(estvarsec/Nsimu);
78 %cfminussec=simulatedvaluesec-1.96*sqrt(estvarsec/Nsimu)
    ;
79 %cfpluspos=simulatedvaluepos+1.96*sqrt(estvarpos/Nsimu);
80 %cfminuspos=simulatedvaluepos-1.96*sqrt(estvarpos/Nsimu)
    ;
81
82 %disp('Confidence interval: ')
83 %disp([cfminusexp,cfplusexp; cfminussec,cfplussec;
        cfminuspos,cfpluspos])
84
85 toc

```