

Interest Rate Theory Solution Sheet 3

1. a) Since the price process of asset S_1 is increasing, we can simply borrow 100\$ to buy the asset S_1 and resell it at time 1. This creates an obvious arbitrage opportunity. However, since the sigma algebra is trivial, every contingent claim must be deterministic. That is, let X denote the payoff of the contingent claim at time T , then $X = a$ a.s., where $a \in \mathbb{R}$. This claim is attainable because we can borrow $a/T * 100$ \$ at time 0 to buy a/T amount of S_1 and sell it at time T such that the payoff of the strategy at time T is $a/T(100 + T - 100) = a$.
- b) Yes. For instance, let's consider the one period Cox-Ross-Rubinstein Model with $u > d > r$, where u (resp. d) denotes the up (resp. down) move for the risky asset, i.e.,

$$S_1 = \begin{cases} S_0(1 + u), & \text{with prob. } p \\ S_0(1 + d), & \text{with prob. } 1-p \end{cases}$$

and r denotes the riskfree interest rate. Since $d > r$, i.e., the risky asset is always outperforming the riskfree asset, the market is not arbitrage free, however it remains complete (cf. MFF lecture notes p. 59).

2. a) First we note that the process

$$Z(t) := \mathcal{E}_t \left(\int_0^t \gamma(s) dW_s \right)$$

is a positive true martingale for $\gamma(t) \equiv -\frac{\mu}{\sigma}$, $t \in [0, T]$, which follows from the Novikov Condition (Theorem 4.7)

$$\mathbb{E}_{\mathbb{P}} \left[e^{1/2 \int_0^T \gamma_s^2 ds} \right] < \infty.$$

Therefore, we can define a probability measure $\mathbb{Q} \sim \mathbb{P}$ with density process given by $Z(t)$

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = Z(t), \quad \text{for all } 0 \leq t \leq T.$$

Bitte wenden!

Then Girsanov's theorem (Theorem 4.6) yields that

$$W^*(t) = W(t) + \frac{\mu}{\sigma}t$$

is a Brownian motion under the measure \mathbb{Q} , and

$$S_1(t) = S_1(0) (1 + \sigma W^*(t)) \quad (1)$$

for the \mathbb{Q} -Brownian motion W^* . Thus \mathbb{Q} is an equivalent martingale measure.

Uniqueness of the martingale measure \mathbb{Q} : The filtration $(\mathcal{F}_t)_{t \in [0, T]}$ was the one generated by the Brownian motion W . Therefore, we are in the situation of Theorem 4.8 (Representation Theorem), thus every probability measure $\hat{\mathbb{Q}}$, which is equivalent to \mathbb{P} has the representation

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \mathcal{E}_\infty(\gamma \bullet W),$$

for some $\gamma \in \mathcal{L}$. However, for the process S_1 to be a martingale under $\hat{\mathbb{Q}}$ the drift of S_1 must vanish, i.e.,

$$S_1(t) = S_1(0) \left(1 + \underbrace{(\mu + \gamma\sigma)t}_{=0} + \underbrace{\sigma(W(t) - \gamma t)}_{\text{is a } \hat{\mathbb{Q}} \text{ BM}} \right).$$

Hence, $\gamma = -\frac{\mu}{\sigma}$ and the equivalent martingale measure is unique.

b) Note that

$$\mathbb{E}_{\mathbb{Q}}[(S_1(T) - K)^+ | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}} \left[\left((S_1(T) - S_1(t) - (K - S_1(t)))^+ \right) | \mathcal{F}_t \right].$$

where

$$K - S_1(t)$$

is \mathcal{F}_t -measurable and

$$S_1(T) - S_1(t) = S_1(0)\sigma(W^*(T) - W^*(t)) \stackrel{d}{=} S_1(0)\sigma W^*(T - t)$$

is independent from \mathcal{F}_t by independence of the increments of the \mathbb{Q} -Brownian motion W^* . Furthermore,

$$S_1(0)\sigma(W^*(T - t)) \sim \mathcal{N}(0, (S_1(0)\sigma)^2(T - t)) \quad (2)$$

under \mathbb{Q} . Thus

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[(S_1(T) - K)^+ | \mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}[(S_1(T) - K)\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] \\ &= \mathbb{E}_{\mathbb{Q}}[(S_1(T) - S_1(t))\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] - \mathbb{E}_{\mathbb{Q}}[(K - S_1(t))\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] \\ &= \mathbb{E}_{\mathbb{Q}}[S_1(0)\sigma(W^*(T) - W^*(t))\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] - (K - S_1(t))\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] \end{aligned}$$

Siehe nächstes Blatt!

where

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{\{S_1(T) - S_1(t) > K - S_1(t)\}} | \mathcal{F}_t] \\ &= \mathbb{Q}[\{S_1(0)\sigma W^*(T-t) > K - x\} | x=S_1(t)]\end{aligned}$$

which by (2) is

$$\begin{aligned}& \int_{K-S_1(t)}^{\infty} \frac{1}{S_1(0)\sigma\sqrt{T-t}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{S_1(0)\sigma\sqrt{T-t}}\right)^2} dx \\ & \stackrel{(*)}{=} \int_{\frac{K-S_1(t)}{S_1(0)\sigma\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \stackrel{(**)}{=} \int_{-\infty}^{\frac{S_1(t)-K}{S_1(0)\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ & = \Phi\left(\frac{S_1(t) - K}{S_1(0)\sigma\sqrt{T-t}}\right)\end{aligned}$$

with the substitution $y = \frac{x}{S_1(0)\sigma\sqrt{T-t}}$ in (*), and using the symmetry $\int_x^{\infty} \varphi(z)dz = \Phi(-x)$ of the standard Gaussian cumulative distribution Φ in (**). Furthermore, by a similar argument as above,

$$\begin{aligned}& \mathbb{E}_{\mathbb{Q}}[S_1(0)\sigma W^*(T-t)\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] \\ & = \mathbb{E}_{\mathbb{Q}}[S_1(0)\sigma W^*(T-t)\mathbb{I}_{\{S_1(0)\sigma W^*(T-t) > K-x\}} | x=S_1(t)] \\ & \stackrel{(2)}{=} \int_{K-S_1(t)}^{\infty} \frac{x}{S_1(0)\sigma\sqrt{T-t}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{S_1(0)\sigma\sqrt{T-t}}\right)^2} dx \\ & \stackrel{(*)}{=} S_1(0)\sigma\sqrt{T-t} \int_{\frac{K-S_1(t)}{S_1(0)\sigma\sqrt{T-t}}}^{\infty} \frac{y}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ & \stackrel{(***)}{=} S_1(0)\sigma\sqrt{T-t} \int_{\frac{K-S_1(t)}{S_1(0)\sigma\sqrt{T-t}}}^{\infty} -\phi'(y) dy \\ & = \left(S_1(0)\sigma\sqrt{T-t}\right) \phi\left(\frac{K-S_1(t)}{S_1(0)\sigma\sqrt{T-t}}\right)\end{aligned}$$

Where we used in (***), that for the standard Gaussian density $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$, we have $\phi'(y) = -y\phi(y)$. Thus we arrived at

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[(S_1(T) - K)^+ | \mathcal{F}_t] &= \\ &= \mathbb{E}_{\mathbb{Q}}[S_1(0)\sigma(W^*(T) - W^*(t))\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] - (K - S_1(t))\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{\{S_1(T) > K\}} | \mathcal{F}_t] \\ &= \left(S_1(0)\sigma\sqrt{T-t}\right) \phi\left(\frac{K-S_1(t)}{S_1(0)\sigma\sqrt{T-t}}\right) + (S_1(t) - K)\Phi\left(\frac{S_1(t) - K}{S_1(0)\sigma\sqrt{T-t}}\right),\end{aligned}$$

as claimed.

Bitte wenden!

3. a) Applying Itô's formula to $f(x) = \|x\|_{\mathbb{R}^d}^2$ and using the Ornstein-Uhlenbeck dynamics we get

$$\begin{aligned} dY(t) &= \left(-\lambda \|X(t)\|^2 + \frac{\sigma^2}{4} d \right) dt + \sigma \sum_{i=1}^d X_i(t) dB_i(t) \\ &= \lambda (\nu - Y(t)) dt + \sigma \sqrt{Y(t)} \sum_{i=1}^d \frac{X_i(t)}{\|X(t)\|} dB_i(t). \end{aligned}$$

To conclude the claim it remains to check that $\sum_{i=1}^d \int_0^t \frac{X_i(s)}{\|X(s)\|} dB_i(s)$ is a standard Brownian motion. Note that $\|X(s)\|$ never hits the origin $\mathbb{P}.a.s.$ and hence the integrands $X_i(t)/\|X(t)\|, i \in \{1, \dots, d\}$ are well-defined, measurable adapted, continuous and bounded. Therefore each stochastic integral is a continuous local martingale. Moreover,

$$\begin{aligned} \left\langle \sum_{i=1}^d \int_0^t \frac{X_i(s)}{\|X(s)\|} dB_i(s), \sum_{i=1}^d \int_0^t \frac{X_i(s)}{\|X(s)\|} dB_i(s) \right\rangle_t &= \int_0^t \sum_{i=1}^d \frac{X_i(s)^2}{\|X(s)\|^2} ds \\ &= \int_0^t ds = t. \end{aligned}$$

Lévy's characterization of Brownian motion now shows that the process Y satisfies

$$Y(t) = \lambda(\nu - Y(t))dt + \sigma\sqrt{Y(t)}dW_t,$$

where W is standard Brownian motion.

- b) See also CIR2.m

4. a) Let $t \in [0, T]$ be arbitrary and let Y be the \mathcal{F}_t -measurable random variable

$$Y = \mathbb{E} \left[\frac{D_T}{D_t} X \mid \mathcal{F}_t \right].$$

Then, for any nonnegative \mathcal{F}_t -measurable random variable Z we have

$$\mathbb{E}_{\mathbb{Q}}[XZ] = \mathbb{E}[D_T X Z] = \mathbb{E} \left[D_t \frac{D_T}{D_t} X Z \right] = \mathbb{E} \left[D_t \mathbb{E} \left[\frac{D_T}{D_t} X \mid \mathcal{F}_t \right] Z \right] = \mathbb{E}_{\mathbb{Q}}[YZ],$$

showing that

$$\mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_t] = Y.$$

Siehe nächstes Blatt!

b) Suppose that M is a \mathbb{Q} -martingale. Then, for all $s \leq t$ we have

$$\mathbb{E}[D_t M_t | \mathcal{F}_s] = D_s \mathbb{E} \left[\frac{D_t}{D_s} M_t | \mathcal{F}_s \right] = D_s \mathbb{E}_{\mathbb{Q}}[M_t | \mathcal{F}_s] = D_s M_s.$$

Conversely, suppose that DM is a \mathbb{P} -martingale. Then, for all $s \leq t$ we have

$$\mathbb{E}_{\mathbb{Q}}[M_t | \mathcal{F}_s] = \frac{1}{D_s} \mathbb{E}[D_t M_t | \mathcal{F}_s] = \frac{1}{D_s} D_s M_s = M_s.$$

5. See `impvol.m`

6. Matlab File

```

1 function sigma=impvol(C,K,T)
2 % EX 3-4) impvol compute the implied volatility sigma
   for a given strike K,
3 % maturity T and value C of an ATM call option at time 0
   in a Bachelier
4 % model
5
6 % at the money
7 s0=K;
8
9 sigma= sqrt(2*pi)*C./(sqrt(T)*s0);
10
11 % More complicated...
12 % Explicit formula for the call option
13 %expcall= @(sigma,Ttemp) s0.*sigma.*sqrt(Ttemp)*normpdf
   ((s0-K)/(s0*sigma*sqrt(Ttemp)))+(s0-K)*normcdf((s0-K)
   /(s0*sigma*sqrt(Ttemp)));
14 % equation satisfied by impvol: C- expcall(sigma)=0
15 %sigma= zeros(1,length(C));
16 % initial value
17 %sigma0=0.16;
18 %for i = 1:length(C)
19 %sigma(i) = fzero(@(sigma) C(i)-expcall(sigma,T(i)),
   sigma0);
20 %end
21
22 end

```

Bitte wenden!

7. Matlab File

```
1 function ex5test
2 % Ex 3–4: plot the implied volatility
3 T=[1/12,2/12,5/12,8/12,11/12,14/12];
4 K= 1125;
5 C=[20.20,30.70,51.00,66.90,81.70,97.00];
6
7 sigma = impvol(C,K,T);
8
9 figure(2)
10 plot(T,sigma,'b')
11 xlabel('time to maturity');
12 ylabel('implied volatility');
13 title('Ex 3–4');
14 end
```

8. Matlab File

```
1 function CIR2
2 % In this exercise we simulated N paths of a CIR process
3 %  $dX_t = \lambda (\nu - X_t) dt + \sigma \sqrt{X_t} dW_t$ ,
    $X_0 = x$  and simulate the
4 % expectation of  $X_1$ ,  $X_1^2$ ,  $X_1^+$  by generating non-
   central  $\chi^2$  random
5 % variables.
6 tic
7 %% parameter input
8 % horizon
9 T=1;
10 % sample size
11 Nsimu=10^4;
12 Nplot=Nsimu;
13 % grid points
14 M=10^3;
15 % volatility
16 lambda=1;
17 nu=1.2;
18 sigma=0.3;
19 x=1;
20 % time step
21 dt= T/M;
22
```

Siehe nächstes Blatt!

```

23 % Check the Feller condition
24 check = 2*lambda*nu >= sigma^2;
25 if check == 1
26     disp('Feller condition is assumed')
27 elseif check==0
28     disp('Feller condition is not satisfied')
29 end
30
31
32 % theoretical value for the expectation , second moment
    and pos part
33 constc = 4*lambda/(sigma^2*(1-exp(-lambda*T)));
34 constv = 4*lambda*nu/(sigma^2);
35 constlambda= constc*x*exp(-lambda*T);
36 % constc_T*X_T is non central chi square distributed (
    constv , constlambda)
37 % E(non-central chisquare distribution) = constv+
    constlambda;
38 %theoreticalvalueexp= (constv+constlambda)/constc;
39 theoreticalvalueexp= x*exp(-lambda*T)+nu*(1-exp(-lambda*
    T));
40 % Var(non-central chisquare distribution) = 2(constv+2*
    constlambda)
41 %theoreticalvaluesec= (2*(constv+2*constlambda)+(constv+
    constlambda)^2)/(constc^2);
42 theoreticalvaluesec= x*sigma^2/lambda*(exp(-lambda*T)-
    exp(-2*lambda*T))+nu*sigma^2/(2*lambda)*(1-exp(-lambda
    *T))^2+theoreticalvalueexp^2;
43 % Use numerical integration to obtain a theoretical
    value
44 % Since the Feller condition is assumed, E[X^+_1] must
    be equal to E[X_1]
45 theoreticalvaluepos=integral(@(x) x.*ncx2pdf(x,constv ,
    constlambda),0,Inf)/constc;
46 %% Simulation
47 % parameters
48 % condition on X_s, c_t*X_t is noncentral chisq
    distributed with dof=
49 % constv and scale parameter = 4
50 % lambda/(sigma^2*(1-exp(-lambda*dt))*X_s*exp(1-lambda*
    dt)
51 dof = constv;

```

Bitte wenden!

```

52 CIR = [x*ones(1,Nplot);zeros(M,Nplot)];
53 for i =1:M
54     CIR(i+1,:)=ncx2rnd(dof*ones(1,Nplot),4*lambda/(sigma
        ^2*(1-exp(-lambda*T/M)))*CIR(i,:)*exp(-lambda*T/M)
        ,[1,Nplot])/(4*lambda/(sigma^2*(1-exp(-lambda*T/M)
        ))));
55 end
56
57 %plot the first 10 sample paths
58 timegrid= 0:dt:T;
59 plot(timegrid,CIR(:,1:10))
60
61 %compute simulated value
62 simulatedvalueexp= mean(CIR(end,:));
63 simulatedvaluesec= mean(CIR(end,:).^2);
64 simulatedvaluepos= mean(subplus(CIR(end,:)));
65
66 disp('Exact values: Expectation/2.Moment/pos. part')
67 disp([theoreticalvalueexp;theoreticalvaluesec;
        theoreticalvaluepos])
68 disp('Estimated value: Expectation/2.Moment/pos. part')
69 disp([simulatedvalueexp;simulatedvaluesec;
        simulatedvaluepos])
70
71 %estimated variance
72 %estvarexp= var(CIR(end,:));
73 %estvarsec= var(CIR(end,:).^2);
74 %estvarpos= var(subplus(CIR(end,:)));
75 % confidence interval using CLT
76 %cfplusexp=simulatedvalueexp+1.96*sqrt(estvarexp/Nsimu);
77 %cfminusexp=simulatedvalueexp-1.96*sqrt(estvarexp/Nsimu)
    ;
78 %cfplussec=simulatedvaluesec+1.96*sqrt(estvarsec/Nsimu);
79 %cfminussec=simulatedvaluesec-1.96*sqrt(estvarsec/Nsimu)
    ;
80 %cfpluspos=simulatedvaluepos+1.96*sqrt(estvarpos/Nsimu);
81 %cfminuspos=simulatedvaluepos-1.96*sqrt(estvarpos/Nsimu)
    ;
82
83 %disp('Confidence interval: ')
84 %disp([cfminusexp,cfplusexp; cfminussec,cfplussec;
        cfminuspos,cfpluspos])

```

Siehe nächstes Blatt!

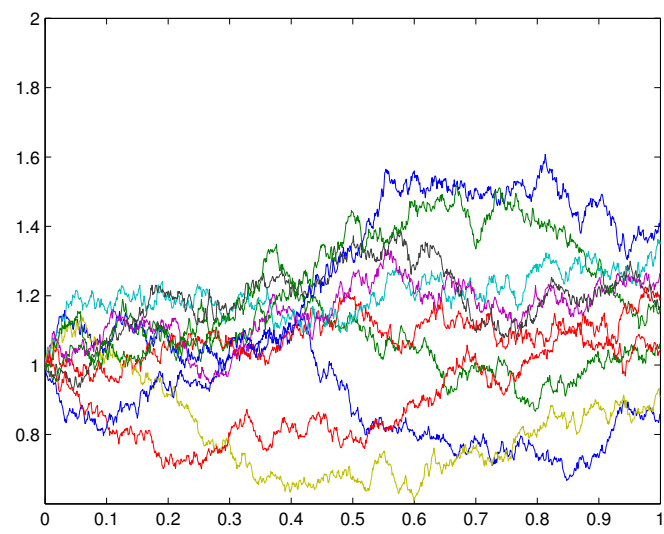


Abbildung 1: 10 sample paths of the CIR process

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86 **toc**