ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Exercise Sheet 1

1. Consider a bond market $(P(t,T))_{(t\leq T)}$ with $P(T,T)=1$ and $P(t,T)>0$ for $t \leq T \leq S$. Let

$$F(t;T,S) := \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$$

denote the simple forward rate and let $F(t,T) := F(t;t,T)$ denote the simple spot rate as in the lecture. Show that

$$P(t,S)F(t;T,S)$$

is the fair value at time t of a contract paying $F(T,S)$ at time S, by constructing a self-financing portfolio with value $P(t,S)F(t;T,S)$ at time t and value $F(T,S)$ at time S.

2. Let us consider an interest rate *cap* which is determined by a number of future dates

$$0 < T_0 < T_1 < \ldots < T_n, \qquad T_i - T_{i-1} = \delta,$$

a fixed *cap rate* $\varkappa > 0$, and a nominal value N which is assumed to be 1. Show that the cash flow of

$$\delta(F(T_{i-1},T_i) - \varkappa)^+$$

at time T_i is equivalent to $(1+\delta\varkappa)$ times the cash flow at date T_{i-1} of a put option on a T_i-bond with strike price $1/(1+\delta\varkappa)$

$$(1+\delta\varkappa)\left(\frac{1}{1+\delta\varkappa} - P(T_{i-1},T_i)\right)^+.$$

3. Let $(\Omega, \mathbb{F}, \mathbb{Q})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let $X = (X_t)_{t\in\mathbb{Z}}$ be an $AR(1)$ process, i.e.,

$$X_t = c + \varphi \cdot X_{t-1} + \varepsilon_t, \tag{1}$$

where $c, \varphi \in \mathbb{R}$ are real constants and ε_t are i.i.d. (and independent of all X_s, $s < t$) Gaussian random variable with zero mean and constant variance σ_ε^2.

Bitte wenden!

a) Show that $AR(1)$ process X is stationary and has finite second moment if and only if $|\varphi| < 1$.

Assume now $|\varphi| < 1$.

b) Verify that

$$\mathbb{E}[X_t] = \frac{c}{1-\varphi}, \qquad Var[X_t] = \frac{\sigma_\varepsilon^2}{1-\varphi^2}.$$

c) Let Y be geometric Brownian motion, i.e.,

$$dY_t = Y_t(\mu dt + \sigma dW_t), \qquad Y_0 = y,$$

where $\mu \in \mathbb{R}, \sigma > 0$ are real constants and W denotes $(\mathbb{Q}, \mathbb{F})$ Brownian motion. Is Y stationary?

4. Incremental simulation of sample path in Matlab Let $T = 1$, $W = (W_t)_{t\geq 0}$ be a $(\mathbb{Q}, \mathbb{F})$ Brownian motion, and $N = (N_t)_{t\geq 0}$ a $(\mathbb{Q}, \mathbb{F})$ Poisson process with intensity parameter $\lambda = 2$. Consider the following processes

(a) **drifted Brownian motion** $X_t^{(a)} = 1 + 2t + 2W_t$

(b) **AR(1) process** $X^{(b)}$ as in previous question with $c = 0.5, \varphi = 0.6$, $\sigma_\varepsilon^2 = 0.2$ and $X_0^{(b)} = 0.1$

(c*) **Poisson Process** $X^{(c)} = N$ with intensity $\lambda = 2$

(d*) **compound Poisson process**

$$X_t^{(d)} = \sum_{i=1}^{N_t} Z_i$$

with i.i.d random variables Z_i. We further assume that

$$Z_i = \begin{cases} 1, & p = 0.5, \\ -1, & p = 0.5. \end{cases}$$

(i) Simulate $N = 10$ sample paths of the processes $X^{(a)}, \cdots, X^{(d)}$ using an equidistant time grid, i.e., $t_i = T \cdot i/M, i = 0, \cdots, M = 10^3$.

(ii) Compute $\mathbb{E}[e^{X_T^{(a)}}]$, $\mathbb{E}[X_T^{(b)}]$, $\mathbb{E}[e^{X_T^{(c)}}]$, $\mathbb{E}[e^{X_T^{(d)}}]$

- explicitly
- using Monte-Carlo simulation with $N = 10^5$ sample paths

Siehe nächstes Blatt!

(iii*) Compute the 95% confident intervals using CLT

*The *-marked questions are primarily meant for students who wish to have stronger and deeper understanding of the subject and are not part of the regular exercises.*

5. Calibrate the forward curve

$$\mathbb{R}_+ \ni x \mapsto f(t_0, t_0 + x) = \phi(x) = \phi(x; z)$$

to a given set of bond prices $P = (p_1, \cdots, p_n)^T$ using the Nelson-Siegel family

$$\phi_{NS}(x; z) = z_1 + (z_2 + z_3 x)e^{-z_4 x}.$$

That is, find $z_* \in \mathbb{R}^4$ such that

$$||P - C \cdot d(z)|| \to \min!$$

where C is the cash flow matrix (cf. Filipovic [Section 3.2.1]) and the implied discount rate is

$$d_i(z) = e^{-\int_0^{T_i} \phi_{NS}(u;z)du}$$

for a payment tenor $0 < T_1 < \cdots < T_N$. Use the numbers in [Filipovic, Table 3.2] (the first three bonds) and plot the calibrated forward curve. Note that the day count convention used is actual/365.
Hint: Use the command *lsqcurvefit* in Matlab with initial value
$z_0 = (0.05, 0.05, 0.05, 0.05)$