ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Solution Sheet 4

1. **a)** By Girsanov's Theorem $W^{\mathbb{Q}}(t) = W(t) - \int_0^t \gamma(s)ds$ is a $\mathbb{Q}-$ Brownian motion, hence

$$\begin{aligned} dr(t) &= b(t)dt + \sigma(t)dW_t \\ &= \left(b(t) + \sigma(t)\gamma(t)\right)dt + \sigma(t)dW^{\mathbb{Q}}(t). \end{aligned}$$

b) By definition the discounted bond price $P(t,T)/B(t)$ is a $\mathbb{Q}-$ martingale. Using the Itô's representation theorem under a change of probability (cf. *Continuous Time Stochastic Control and Optimization with Financial Application, Huyen Pham* Theorem 1.2.14), there exists a function $\psi(\cdot,T) \in \mathcal{L}$ such that

$$\frac{P(t,T)}{B(t)} = P(0,T) + \int_0^t \psi(s)dW^{\mathbb{Q}}(s).$$

To get the desired dynamics of $P(t,T)$ we again apply Itô's formula

$$dP(t,T) = d\left(B(t) \cdot \frac{P(t,T)}{B(t)}\right) = P(t,T)r(t)dt + \psi(t)B(t)dW^{\mathbb{Q}}(t).$$

Since the bond price $P(t,T) > 0$ we can define $v(t,T) := \psi(t)B(t)/P(t,T)$ and get

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + v(t,T)dW^{\mathbb{Q}}(t).$$

c) Again using Itô's formula we get

$$\begin{aligned} d\left(\frac{P(t,T)}{B(t)}\right) &= \frac{1}{B(t)}dP(t,T) - \frac{P(t,T)}{B(t)^2}dB(t) \\ &= \left(\frac{P(t,T)}{B(t)}\right)\big(v(\cdot,T)dW^{\mathbb{Q}}(t) + r(t)dt - r(t)dt\big) \\ &= \left(\frac{P(t,T)}{B(t)}\right)v(\cdot,T)dW^{\mathbb{Q}}(t). \end{aligned}$$

Put differently,

$$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}\left(v(\cdot,T) \bullet W^{\mathbb{Q}}\right)_t.$$

Bitte wenden!

2. Note that by definition it holds

$$r(s) = r(t)e^{\beta(s-t)} + \frac{b}{\beta}(e^{\beta(s-t)} - 1) + \sigma e^{\beta s}\int_t^s e^{-\beta u} dW(u), \quad \text{for} \quad 0 \leq t < s. \quad (1)$$

In order to show that the integral is normally distributed, the idea is to use the stochastic Fubini theorem[1] and apply Ex 2-4, i.e.,

$$\int_0^T f(s)dW(s) \sim \mathcal{N}\big(0, \int_0^T (f(s))^2 ds\big).$$

for any $T > 0$ and any deterministic function

$$f : [0, T] \longrightarrow \mathbb{R} \in C^0([0, T]).$$

In concrete terms,

$$\begin{aligned}
\int_t^T r(s)ds &= r(t)\int_t^T e^{\beta(s-t)}ds + \frac{b}{\beta}\int_t^T (e^{\beta(s-t)} - 1)ds + \sigma \int_t^T \int_t^s e^{\beta(s-u)}dW(u)ds, \\
&= \frac{r(t)}{\beta}(e^{\beta(T-t)} - 1) + \frac{b}{\beta^2}(e^{\beta(T-t)} - 1) - \frac{b(T-t)}{\beta} + \sigma \int_t^T \int_u^T e^{\beta(s-u)}dsdW(u), \\
&= \frac{r(t)}{\beta}(e^{\beta(T-t)} - 1) + \frac{b}{\beta^2}(e^{\beta(T-t)} - 1 - \beta(T-t)) + \frac{\sigma}{\beta}\int_t^T (e^{\beta(T-u)} - 1)dW(u).
\end{aligned}$$

The first two terms on the right hand side are $\mathcal{F}_t$ measurable, while the last term is independent of $\mathcal{F}_t$ and is normally distributed with mean 0 and variance Σ_t. To find the variance Σ_t^2 we use Itô's Isometry

$$\begin{aligned}
\Sigma_t^2 &= \frac{\sigma^2}{\beta^2}\int_t^T (e^{\beta(T-t)-1})^2 ds \\
&= \frac{\sigma^2}{\beta^2}\int_t^T (e^{2\beta(T-t)-1} - 2e^{\beta(T-s)} + 1)ds \\
&= \frac{\sigma^2}{\beta^2}\left(-\frac{1}{2\beta}(1 - e^{2\beta(T-t)}) + \frac{2}{\beta}(1 - e^{\beta(T-t)}) + (T-t)\right) \\
&= \frac{\sigma^2(-4e^{\beta(T-t)} + e^{2\beta(T-t)} + 2\beta(T-t) + 3)}{2\beta^3}.
\end{aligned}$$

[1]The stochastic Fubini Theorem is applicable, since

$$\int_t^T \left(\int_t^T |e^{\beta(s-u)}|^2 du\right)^{1/2} ds < \infty \quad \text{a.s.}$$

Siehe nächstes Blatt!

Putting everything together, we get

$$
\begin{aligned}
P(t,T) &= \mathbb{E}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[e^{-\left(\frac{r(t)}{\beta}(e^{\beta(T-t)}-1)+\frac{b}{\beta^2}(e^{\beta(T-t)}-1-\beta(T-t))\right)} e^{-\frac{\sigma}{\beta}\int_t^T (e^{\beta(T-u)}-1)dW(u)} \middle| \mathcal{F}_t \right] \\
&= e^{-\left(\frac{r(t)}{\beta}(e^{\beta(T-t)}-1)+\frac{b}{\beta^2}(e^{\beta(T-t)}-1-\beta(T-t))\right)} e^{\frac{1}{2}\Sigma_t^2} \\
&= \exp\Bigg(-\frac{r(t)}{\beta}\left(e^{\beta(T-t)}-1\right) - \frac{b}{\beta^2}\left(e^{\beta(T-t)}-1-\beta(T-t)\right) \\
&\quad + \frac{\sigma^2}{4\beta^3}\left(e^{2\beta(T-t)} - 4e^{\beta(T-t)} + 2\beta(T-t) + 3\right) \Bigg).
\end{aligned}
$$

3. **a)** Applying Itô's formula to $M(t) = F(t,r(t))e^{-\int_0^t r(s)ds}$ and setting the drift equal to zero, the term-structure equation associated to the Vasiček model is

$$\partial_t F(t,r) + (b+\beta r(t))\,\partial_r F(t,r) + \frac{1}{2}\sigma^2 \partial_{rr} F(t,r) - r(t)F(t,r) = 0. \qquad (2)$$

b) Since the drift-, and diffusion terms of the driving Ornstein-Uhlenbeck process in the Vasiček model are of the affine form as in (5.7) of Proposition 5.2, it is a reasonable guess that the Vasiček model provides an affine term structure. Therefore, we assume that there exist smooth deterministic functions A and B such that

$$F(t,r;T) \equiv e^{-A(t,T)-B(t,T)r}. \qquad (3)$$

This guess is verified, if it is possible to choose these functions A and B such that they satisfy[2]

$$
\begin{aligned}
\partial_t A(t,T) =& \frac{1}{2}\sigma^2 B^2(t,T) - bB(t,T), \qquad (4) \\
\partial_t B(t,T) =& -\beta B(t,T) - 1, \qquad (5)
\end{aligned}
$$

with terminal condition

$$A(T,T) = 0 \quad \text{and} \quad B(T,T) = 0,$$

since then Proposition 5.2 yields that the associated model indeed provides an affine term structure. By (3), the term-structure equation (2) becomes

$$
\begin{aligned}
F(t,r)\left(\partial_t A(t,T) - r(t)\partial_t B(t,T)\right) - (b+\beta r(t))\,F(t,r)B(t,T) \\
+\frac{1}{2}\sigma^2 F(t,r)B(t,T)^2 - F(t,r)r(t) = 0
\end{aligned}
$$

[2](5.8) and (5.9) of Proposition 5.2 for the Vasiček model.

for any $0 \leq t \leq T$ and $r(t)$. Since $F(t,r) > 0$, we can simplify the above to

$$\left(\partial_t A(t,T) - bB(t,T) + \tfrac{1}{2}\sigma^2 B(t,T)^2\right) + r(t)\left(\partial_t B(t,T) - \beta B(t,T) - 1\right) = 0.$$

The above holds for all $0 \leq t \leq T$ and $r(t)$ if and only if A and B satisfy both equations (4) and (5). Equation (5) yields

$$B(t,T) = \frac{1}{\beta}\left(e^{\beta(T-t)} - 1\right), \tag{6}$$

where the terminal condition $B(T,T) = 0$ holds. From equation (5) we find

$$\begin{aligned} A(T,T) - A(t,T) &= \int_t^T \partial_s A(s,T)ds \\ &= \int_t^T \left(\frac{1}{2}\sigma^2 B^2(s,T) - bB(s,T)\right) ds. \end{aligned}$$

Using the expression (6) for the integrand at $t \leq s < T$, the above becomes

$$\begin{aligned} &\frac{1}{2}\sigma^2 \int_t^T B^2(t,T)ds - b\int_t^T B(t,T)ds \\ =&\frac{\sigma^2}{4\beta^3}\left(e^{2\beta(T-t)} - 1 - 4e^{\beta(T-t)} + 4 + 2\beta(T-t)\right) - \frac{b}{\beta^2}\left(e^{\beta(T-t)} - 1 - \beta(T-t)\right). \end{aligned}$$

Together with the terminal condition $A(T,T) = 0$, which follows from (6) we obtain

$$\begin{aligned} A(t,T) = &-\frac{\sigma^2}{4\beta^3}\left(e^{2\beta(T-t)} - 4e^{\beta(T-t)} + 2\beta(T-t) + 3\right) \\ &+ \frac{b}{\beta^2}\left(e^{\beta(T-t)} - 1 - \beta(T-t)\right). \end{aligned} \tag{7}$$

Hence, in fact the Vasiček model provides an affine term stucture (3), with functions B and A as in (6) and (7).

c) To verify the martingale property we check that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T |\partial_r F(t,r(t))e^{-\int_0^t r(u)du}\sigma(t,r(t))|^2 dt\right] < \infty.$$

Inserting the affine solution (2) we have to verify that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T |-B(t,T)e^{-A(t,T)-B(t,T)r(t)}e^{-\int_0^t r(u)du}\sigma|^2 dt\right] < \infty.$$

Since the functions $A(\cdot,T)$ and $B(\cdot,T)$ are both continuous and hence bounded on the compact interval $[0,T]$, up to a constant term we have to show that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-2\cdot B(t,T)r(t)}e^{-2\int_0^t r(u)du}dt\right] < \infty. \tag{8}$$

Recall that $r(t)$ is OU process and hence is normally distributed (cf. Ex 2-4). We denote its mean and variance by $\tilde{\mu}(t)$ and $\tilde{\sigma}^2(t)$, respectively. Moreover, in Ex 4-2 we have shown that $\int_0^t r(u)du$ is also normally distributed and let us denote its mean and variance by $\mu(t)$ and $\sigma^2(t)$, respectively. Consequently, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{kr(t)}\right] &= e^{k\tilde{\mu}(t)+\frac{k^2}{2}\tilde{\sigma}^2(t)}, \quad \text{for} \quad k \in \mathbb{R},\\
\mathbb{E}^{\mathbb{Q}}\left[e^{k\int_0^t r(u)du}\right] &= e^{k\mu(t)+\frac{k^2}{2}\sigma^2(t)}, \quad \text{for} \quad k \in \mathbb{R}.
\end{aligned}$$

Now, applying Tonelli's theorem and Cauchy Schwarz inequality we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-2B(t,T)r(t)}e^{-2\int_0^t r(u)du}dt\right] &= \int_0^T \mathbb{E}^{\mathbb{Q}}\left[e^{-2B(t,T)r(t)}e^{-2\int_0^t r(u)du}\right]dt\\
&\le \int_0^T \left(\mathbb{E}^{\mathbb{Q}}\left[e^{-4B(t,T)r(t)}\right]\mathbb{E}^{\mathbb{Q}}\left[e^{-4\int_0^t r(u)du}\right]\right)^{1/2}dt\\
&= \int_0^T e^{\frac{1}{2}\left(-4B(t,T)\tilde{\mu}(t)+8B(t,T)^2\tilde{\sigma}^2(t)-4\mu(t)+8\sigma^2(t)\right)}dt\\
&< \infty,
\end{aligned}$$

since the functions $\tilde{\mu}(t), \tilde{\sigma}(t), \mu(t), \sigma(t)$ are all continuous.

4. Matlab File

```
function [valuepde, valuedis, valuemc]=bondvasicek
% In this exercise we compute the bond price at time 0 in a Vasicek
% shortrate model
tic
%% parameter input
% horizon
T=10;
% sample size
Nsimu=10^5;
Nplot=Nsimu;
% grid points
M=10^3;
% volatility
```

Bitte wenden!

```
sigma=0.04;
beta=-0.86;
b=0.08;
% mean reversion speed = lambda=-beta
lambda=-beta;
% mean reversion level = nu=-b/beta
nu=-b/beta;
% initial value = r_0
r0=0.08;
% time step
dt= T/M;

%% Analytical approach
AT= sigma^2*(4*exp(beta*T)-exp(2*beta*T)-2*beta*T-3)/(4*beta^3)+b*(exp(beta*T)-1-beta*T)/(beta^2);
BT= (exp(beta*T)-1)/beta;
valuepde=exp(-AT-BT*r0);

%% Distribution approach
mu0= r0/beta*(exp(beta*T)-1)+b/(beta^2)*(exp(beta*T)-1-beta*T);
sigma0sq= sigma^2/(2*beta^3)*(-4*exp(beta*T)+exp(2*beta*T)+2*beta*T+3);

% generate Nsimu*standard normal random variables
stdnormal = randn(1,Nsimu);
valuedis= mean(exp(-mu0+sqrt(sigma0sq)*stdnormal));


%% Monte Carlo approach
% BM
BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
OU = [r0*ones(1,Nplot);zeros(M,Nplot)];
% Euler Maruyama Scheme
for i =1:M
    OU(i+1,:)=OU(i,:)+lambda*(nu-OU(i,:))*dt+ sigma.*(BM(i+1,:)-BM(i,:));
end

integral = zeros(1,Nsimu);
intgrid= 0:dt:T;
for j= 1:Nsimu
```

Siehe nächstes Blatt!

```
    % use trapezoidal rule
    integral(j)= trapz(intgrid,OU(:,j));
end
% take the expectation
valuemc= mean(exp(-integral));
toc
```