ETH Zürich HS 2014
D-MATH
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Interest Rate Theory
Solution Sheet 5

1. **a)** Since the process $(r_t)_{t\geq 0}$ satisfies the equation

$$r(t) = r_0 + \int_0^t (a - br(s))ds + \sigma \int_0^t \sqrt{r(s)}dW_s, \quad t \geq 0, \tag{1}$$

the conditional probability distribution of r (cf. Ex 3-3) is known and hence expectation can be computed by a straightforward integration. Here, we also consider an alternative method by showing that the stochastic integral is a true martingale.[1] We verify that

$$\mathbb{E}\left[\int_0^T r(s)ds\right] = \int_0^T \mathbb{E}[r(s)]ds < \infty, \tag{2}$$

where we have used Tonelli's Theorem in the first step. Indeed, given r_0 the random variable $c_t r(t)$ follows a non-central χ^2- distribution with degree of freedom $k := 4a/\sigma^2$ and non-centrality parameter $\lambda(t) := c_t r_0 e^{-at/b}$ with $c_t := 4b/(\sigma^2(1 - e^{-at/b}))$. Since the function c is continuous and the distribution is χ^2 we note that $s \mapsto \mathbb{E}[r(s)]$ is also continuous and hence (2) is satisfied. Now, taking expectation in (1) yields

$$\mathbb{E}[r(t)] = r_0 + \int_0^t (a - b\mathbb{E}[r(s)])ds.$$

Therefore, the function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ given by $\varphi(t) = \mathbb{E}[r(t)]$ satisfies the ordinary differential equation

$$\begin{cases} \varphi'(t) &= a - b\varphi(t), \\ \varphi(0) &= r_0. \end{cases}$$

[1]This alternative method might be quite useful if the distribution of the solution of certain SDE is not easy to compute (e.g., Shiryaev-Roberts SDE). Then, some other tricks might be needed to show the integrability condition (2).

The solution of this ordinary differential equation is given by

$$\varphi(t) = r_0 e^{-bt} + a \int_0^t e^{-b(t-s)} ds = r_0 e^{-bt} + \frac{a}{b}\big(1 - e^{-bt}\big).$$

Hence,

$$\lim_{t\to\infty} \mathbb{E}[r(t)] = \frac{a}{b}.$$

b) Using Itô's formula, we find that the process

$$r(t) = r_0 e^{(\beta - \sigma^2/2)t + \sigma W(t)}, \quad t \geq 0$$

is a solution for the stochastic differential equation

$$r(t) = r_0 + \beta \int_0^t r(s) ds + \sigma \int_0^t r_s dW(s)$$

with initial value $r_0 \in \mathbb{R}$ and constants $\beta, \sigma \in \mathbb{R}$. Therefore,

$$\begin{aligned} \mathbb{E}[r(t)] &= \mathbb{E}\big[r_0 e^{(\beta-\sigma^2/2)t + \sigma W(t)}\big] \\ &= r_0 e^{\beta t} \mathbb{E}\big[e^{-(\sigma^2/2)(t) + \sigma W(t)}\big] \\ &= r_0 e^{\beta t}. \end{aligned}$$

As $t \to \infty$, we have the following asymptotic behavior:

$$\mathbb{E}[r_t] \begin{cases} \to 0, & \text{for} \quad \beta < 0, \\ \to r_0, & \text{for} \quad \beta = 0, \\ \text{diverges}\ , & \text{for} \quad \beta > 0. \end{cases}$$

2. **a)** By Proposition 6.1 the short rate process related to the HJM forward rate dynamics

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T) ds + \int_0^t \sigma(s,T) dW^*(s)$$

is given by

$$r(t) = r(0) + \int_0^t \zeta(u) du + \int_0^t \sigma(u,u) dW^*(u), \tag{3}$$

where

$$\zeta(u) = \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u) ds + \int_0^u \partial_u \sigma(s,u) dW^*(s). \tag{4}$$

Siehe nächstes Blatt!

The short rate dynamics in the Hull-White-extended Vasiček model

$$r(t) = r(0) + \int_0^t (b(u) + \beta r(u))du + \int_0^t \sigma dW^*(t) \tag{5}$$

a priori only determine the volatility process of the short rate process related to the given HJM forward rate dynamics to be $\sigma(u,u) \equiv \sigma$. We claim that the general HJM forward rate volatility is given by $\sigma(t,T) = \sigma e^{\beta(T-t)}$ for all $0 \leq t \leq T$. This can be verified as follows: In general, it holds by definition[2] of the instantaneous forward rate that

$$f(t,T) = -\frac{\partial}{\partial T}\log(P(t,T)), \quad \text{for all} \quad 0 \leq t \leq T. \tag{6}$$

Furthermore, we know from the lecture, that the model provides an affine term-structure, therefore we have

$$P(t,T) = \exp(-A(t,T) - B(t,T)r(t)) \tag{7}$$

for the bond prices, where the functions A and B are known, and combining (6) and (7) we find

$$f(t,T) = \partial_T A(t,T) + \partial_T B(t,T)r(t), \quad \text{for all} \quad 0 \leq t \leq T.$$

With

$$B(t,T) = \frac{1}{\beta}(e^{\beta(T-t)} - 1) \tag{8}$$

thus,

$$\partial_T B(t,T) = e^{\beta(T-t)}, \tag{9}$$

it follows in particular, that we have an expression of the form[3]

$$f(t,T) = (\dots) + e^{\beta(T-t)}r(t) \tag{10}$$

for the forward curve related to $r(t)$, where the term $(\dots)$ is of finite variation. Applying Itô's formula to the function $f(x,t) = xe^t$, yields that the explicit solution to the stochastic differential equation (5) is

$$r(t) = r(0)e^{\beta t} + \int_0^t e^{-\beta(s-t)}b(s)ds + \int_0^t \sigma e^{-\beta(s-t)}dW^*(s).$$

Together with (10) we find that

$$\sigma(s,T) = \sigma e^{\beta(T-t)}e^{-\beta(s-t)} = \sigma e^{\beta(T-s)}, \quad \text{for all} \quad 0 \leq s \leq T, \tag{11}$$

[2] See: equation (2.1) in Filipović, T.S.M.

[3] See also: equation (5.12) of Filipović, T.S.M.

Bitte wenden!

as claimed. Under the assumption that the HJM drift condition is satisfed, (11) already determines the drift term (4) in the short rate process (3):
If the HJM drift condition is satisfed, then

$$\begin{aligned}\alpha(s,T) &= \sigma(s,T)\int_s^T \sigma(s,u)du, \quad \text{for all} \quad 0\le s\le T\\ &\overset{(11)}{=} \sigma^2 e^{\beta(T-s)}\int_s^T e^{\beta(u-s)}du,\\ &= \sigma^2\frac{(e^{2\beta(T-s)}-e^{\beta(T-s)})}{\beta}.\end{aligned}$$

and thus the drift-term (4) becomes

$$\zeta(u) = \partial_u f(0,u) + \int_0^u \left(2\sigma^2 e^{2\beta(u-s)} - \sigma^2 e^{\beta(u-s)}\right) ds + \beta\int_0^u \sigma e^{\beta(u-s)} dW^*(s). \tag{12}$$

Now by definition $r(t) = f(t,t)$, therefore

$$\begin{aligned}r(u) = f(u,u) &= f(0,u) + \int_0^u \alpha(s,u)ds + \int_0^u \sigma(s,u)dW^*(s)\\ &= f(0,u) + \int_0^u \sigma^2\frac{(e^{2\beta(u-s)}-e^{\beta(u-s)})}{\beta}ds + \int_0^u \sigma e^{\beta(u-t)}dW^*(s),\end{aligned}$$

therefore,

$$\beta\int_0^u \sigma e^{u-s}dW^*(s) = \beta r(u) - \beta f(0,u) - \int_0^u \sigma^2(e^{2\beta(u-s)} - e^{\beta(u-s)})ds. \tag{13}$$

Combining (13) and (12) yields

$$\begin{aligned}b(u) = \zeta(u) - \beta r(u) &= \partial_u f(0,u) + \int_0^u \sigma^2 e^{2\beta(u-s)}ds - \beta f(0,u)\\ &= \partial_u f(0,u) - \beta f(0,u) + \frac{\sigma^2}{2\beta}(e^{2\beta u}-1), \quad \text{for all} \quad u\ge 0.\end{aligned} \tag{14}$$

Hence by choosing the initial forward curve to be $f^*(0,u) = -\frac{\partial}{\partial u}\log(P^*(0,u))$, the Hull-White extended Vasiček model with $b(u)$ as in (14), gives a perfect fit to the initial bond curve $P^*(0,u)$, $u\ge 0$.

b) Let $T\mapsto f^*(0,T)$ be the initial forward rate curve

$$f^*(0,T) = -\partial_T \log P^*(0,T).$$

Siehe nächstes Blatt!

In order to fit the initial bond prices, we must have

$$f^*(0,T) = \partial_T A(0,T) + \partial_T B(0,T) r_0.$$

We have seen in the lecture that the correponding functions in the affine term-structure are

$$\begin{aligned} A(0,T) &= -\frac{\sigma^2}{2}\int_0^T B^2(s,T)ds + \int_0^T b(s)B(s,T)ds, \\ B(0,T) &= \frac{1}{\beta}(e^{\beta T}-1), \end{aligned}$$

with derivatives $\partial_T B(0,T) = e^{\beta T}$ and

$$\begin{aligned} &\partial_T A(0,T) = -\frac{\sigma^2}{2}\int_0^T \frac{\partial}{\partial T} B^2(s,T)ds + \int_0^T b(s)\partial_T B(s,T)ds \\ &= \frac{\sigma^2}{2}\int_0^T \frac{\partial}{\partial s} B^2(s,T)ds + \int_0^T b(s)e^{\beta(T-s)}ds = -\frac{\sigma^2}{2}B^2(0,T) + \int_0^T b(s)e^{\beta(T-s)}ds \\ &= -\frac{\sigma^2}{2\beta^2}(e^{\beta T}-1)^2 + \int_0^T b(s)e^{\beta(T-s)}ds. \end{aligned}$$

Therefore, defining the functions

$$\begin{aligned} g(T) &:= \frac{\sigma^2}{2\beta^2}(e^{\beta T}-1)^2, \\ \phi(T) &:= \int_0^T b(s)e^{\beta(T-s)}ds + e^{\beta T} r_0 \end{aligned}$$

we arrive at

$$f^*(0,T) = -g(T) + \phi(T).$$

The function ϕ satisfies the ordinary differential equation

$$\begin{cases} \partial_T \phi(T) &= \beta\phi(T) + b(T) \\ \phi(0) &= r_0. \end{cases}$$

It follows that

$$\begin{aligned} b(T) &= \partial_T \phi(T) - \beta\phi(T) \\ &= \partial_T (f^*(0,T) + g(T)) - \beta(f^*(0,T) + g(T)) \\ &= \partial_T f^*(0,T) - \beta f^*(0,T) + \frac{\sigma^2}{2\beta}(e^{2\beta t}-1). \end{aligned}$$

Bitte wenden!

3. a)

$$\begin{aligned}
P(0,1) &= e^{-f(0,1)} = e^{-0.04},\\
P(0,2) &= e^{-f(0,1)-f(0,2)} = e^{-0.08},\\
P(0,3) &= e^{-f(0,1)-f(0,2)-f(0,3)} = e^{-0.12}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(1,1)(\omega_1) &= 1,\\
P(1,2)(\omega_1) &= e^{-0.06},\\
P(1,3)(\omega_1) &= e^{-0.12},
\end{aligned}$$

and

$$\begin{aligned}
P(1,1)(\omega_2) &= 1,\\
P(1,2)(\omega_2) &= e^{-0.02},\\
P(1,3)(\omega_2) &= e^{-0.04}.
\end{aligned}$$

The matrix

$$A = \begin{pmatrix} P(0,1) & P(0,2) & P(0,3) \\ P(1,1)(\omega_1) & P(1,2)(\omega_1) & P(1,3)(\omega_1) \\ P(1,1)(\omega_2) & P(1,2)(\omega_2) & P(1,3)(\omega_2) \end{pmatrix} = \begin{pmatrix} e^{-0.04} & e^{-0.08} & e^{-0.12} \\ 1 & e^{-0.06} & e^{-0.12} \\ 1 & e^{-0.02} & e^{-0.04} \end{pmatrix}$$

resulting from these values has the nonvanishing determinant

$$\begin{aligned}
\det(A) =& e^{-0.04}e^{-0.06}e^{-0.04} + e^{-0.08}e^{-0.12} + e^{-0.02}e^{-0.12}\\
&- e^{-0.06}e^{-0.12} - e^{-0.02}e^{-0.12}e^{-0.04} - e^{-0.08}e^{-0.04} \neq 0.
\end{aligned}$$

Therefore, the matrix A is invertible.

b) The arbitrage strategy we seek, is a predictable process $\phi = (\phi^1(n), \phi^2(n), \phi^3(n))_{n=1,2}$, such that the following requirements hold

$$V(0) = 0 \qquad (*)$$

$$V(1)(\omega_1) = \sum_{i=1}^{3} \phi^1(i)(P(1,i))(\omega_1) = 1 \qquad (**)$$

$$V(1)(\omega_2) = \sum_{i=1}^{3} \phi^1(i)(P(1,i))(\omega_2) = 1 \qquad (***)$$

Siehe nächstes Blatt!

By the self-financing condition (s-f.c.), we furthermore have on ϕ the requirement

$$V(1) - V(0) = \sum_{i=1}^{3} \phi^1(i)(P(1,i) - P(0,i)). \tag{s-f.c.}$$

The self-financig condition together with $(*)$ yields

$$\sum_{i=1}^{3} \phi^1(i)(P(0,i)) = 0. \tag{****}$$

Therefore, it remains to find a vector $\phi(1) = (\phi^1(1), \phi^2(1), \phi^3(1))^\top$, such that

$$\begin{pmatrix} P(0,1) & P(0,2) & P(0,3) \\ P(1,1)(\omega_1) & P(1,2)(\omega_1) & P(1,3)(\omega_1) \\ P(1,1)(\omega_2) & P(1,2)(\omega_2) & P(1,3)(\omega_2) \end{pmatrix} \begin{pmatrix} \phi^1(1) \\ \phi^2(1) \\ \phi^3(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

In this system of equations the third and second lines follow from (***) and (**), and the first line follows from (****). Since by part a), the matrix A is invertible, the above vector is given by

$$\begin{pmatrix} \phi^1(1) \\ \phi^2(1) \\ \phi^3(1) \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

4. **a)** In the general HJM setup we have

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s), \quad t \leq T$$

for a given initial forward curve $T \mapsto f(0,T)$. The HJM drift condition

$$b(t,T) = -v(t,T)\gamma(t) \quad \text{for all } T,\ d\mathbb{P} \otimes dt - a.s., \tag{15}$$

where

$$v(t,T) := -\int_t^T \sigma(t,u)du,$$

and

$$b(t,T) := -\int_t^T \alpha(t,u)du + \frac{1}{2}||v(t,T)||^2.$$

Bitte wenden!

If the HJM drift condition (15) is satisfied, the dynamics of the forward rate under the equivalent local martingale measure $\mathbb{Q}$ become:

$$f(t,T) = f(0,T) + \int_0^T \left(\sigma(s,T) \int_s^T \sigma(s,u)du \right) ds + \int_0^t \sigma(s,T)dW^*(s). \tag{16}$$

In our current setting we have

$$\begin{array}{lll} f(t,T) = & h(T-t) + \int_0^t b(s)ds + \int_0^t \varrho(s)dW^*(s), & 0 < t \leq T \\ f(0,T) = & h(T) & t = 0. \end{array}$$

Therefore, $\sigma(s,T) \equiv \varrho(s)$. Assuming that the HJM drift condition holds, (16) yields

$$f(0,T) + \int_0^t \varrho^2(s)(T-s)ds = h(T-t) + \int_0^t b(s)ds.$$

Solving for h, the above becomes

$$h(T-t) = f(0,T) + \int_0^t \varrho^2(s)(T-s)ds - \int_0^t b(s)ds. \tag{17}$$

Taking the derivative with respect to t on both sides gives

$$-h'(T-t) = \varrho^2(t)(T-t) - b(t), \quad \text{for all } t, T \quad \text{with } t \leq T, \tag{18}$$

and taking the derivative of (18) with respect to T on both sides yields

$$-h''(T-t) = \varrho^2(t), \quad \text{for all } t, T \quad \text{with } t \leq T. \tag{19}$$

Since the left hand side of (19) only depends on the difference $T-t$ and not on T and t separately, and since the right hand side of (19) is independent of T, it follows that equation (19) can only hold for all t, T with $t \leq T$ if ϱ^2 is constant. Hence, $\varrho^2(t) \equiv a$. Inserting this into (18) we find

$$-h'(T-t) - a(T-t) = b(t), \quad \text{for all } t, T \quad \text{with } t \leq T,$$

and the same argument as above yields that $b(t) \equiv b$ is also constant. Integrating $-h'(T-t) = a(T-t) + b$ with respcet to t and using (17) with $f(0,T) \equiv h(T)$ yields

$$\begin{aligned} h(T-t) - h(T) &= \int_0^t -h'(T-s)ds \\ &= \int_0^t a(T-s) - b \, ds \\ &= (aT-b)t - a\frac{t^2}{2}. \end{aligned}$$

Siehe nächstes Blatt!

The requirement

$$h(T-t) = h(T) + (aT-b)t - a\frac{t^2}{2}, \quad \text{for all } t, T \quad \text{with } t \le T \tag{20}$$

implies that the right hand side of (20) may only depend on the difference $T-t$ and not on T and t separately. This determines the function $h(T)$ as follows: $h(T) = \widetilde{h}(T) + \frac{a}{2}T^2 - bT$ for some appropriately chosen function $\widetilde{h}$. Indeed in this case, the right hand side of (20) is of the form

$$\begin{aligned} h(T-t) =& h(T) + aTt - bt - a\frac{t^2}{2} \\ =& \widetilde{h}(T) + \frac{a}{2}T^2 - bT + \frac{a}{2}2Tt - bt - \frac{a}{2}t^2 \\ =& \widetilde{h}(T) - \frac{a}{2}(T-t)^2 + b(T-t), \end{aligned}$$

and along the lines of the previous arguments, since $\widetilde{h}(T)$ does not depend on t, it has to be a constant $\widetilde{h}(T) \equiv c$. Therefore, $h(x) \equiv c - \frac{a}{2}x^2 + bx$, for all $x \ge 0$.

b) In the discussion of short-rate models[4], we fixed a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{Q})$, where we assumed $\mathbb{Q}$ to be a martingale measure and denoted by W^* a one-dimensional $\mathbb{Q}$-Brownian motion. We have shown above that the HJM drift condition implies that under $\mathbb{Q}$ the forward curve evolution is

$$\begin{array}{lll} f(t,T) = & f(0,T) + at\left(T - \frac{t}{2}\right) + \int_0^t \sqrt{a} dW^*(s), & 0 < t \le T, \\ f(0,T) = & h(T) & t = 0. \end{array}$$

The associated short rate process is given by

$$r(t) = f(t,t) = f(0,t) + \frac{at^2}{2} + \sqrt{a}W^*(t).$$

Setting $a = \sigma^2$ the above corresponds to the Ho-Lee model, which is described by the dynamics

$$r(t) = \int_0^t b(s)ds + \sigma W^*(t)$$

with

$$\int_0^t b(s)ds = f(0,t) + \frac{\sigma^2 t^2}{2}. \tag{21}$$

In fact, this confirms the results derived in the lecture, where it was shown[5] that with the choice of parameters as in (21), the Ho-Lee model indeed gives a perfect fit to the observed initial forward curve $f(0,T)$.

[4]See section *5.2 Diffusion Short-Rate Models*, p. 80 ff. of Filipović, T.S.M.

[5]See also Section 5.4.4 p.89 ff. of Filipović, T.S.M.

Bitte wenden!

c) By assumption, the initial forward curve is described by $f(0,T) \equiv h(T)$ for a deterministic function $T \mapsto h(T)$, and we have shown above, that if the forward curve evolution is given by parallel shifts $f(t,T) = h(T-t) + Z(t)$, then the HJM drift condition already determines the form of h to be

$$h(T) \equiv -\frac{a}{2}T^2 + bT + c$$

for some appropriately chosen constants a, b and c. A generic initial forward curve $f(0,T)$ is an arbitrary function, in particular it is not necessarily of second-order-polynomial form. If this is not the case, $f(0,T) \equiv h(T)$ leads to a contradiction.

5. Matlab File

```
function [valuepde,valuemc]=bondCIR
% In this exercise we compute the bond price at time 0 in a CIR shortrate model dr_t= (a-b
% r_t) dt + \sigma \sqrt(r_t) dW_t
tic
%% parameter input
% horizon
T=10;
% sample size
Nsimu=10^5;
Nplot=Nsimu;
% grid points
M=10^3;
% volatility
sigma=0.04;
a=0.0052;
b=0.0447;
% initial value = r_0
r0=0.08;
% time step
dt= T/M;

%% Analytical approach
gamma = sqrt(b^2+2*sigma^2);
AT= -2*a/(sigma^2)*log(2*gamma*exp((gamma+b)*T/2)/((gamma+b)*(exp(gamma*T)-1)+2*gamma));
BT= 2*(exp(gamma*T)-1)/((gamma+b)*(exp(gamma*T)-1)+2*gamma);
valuepde=exp(-AT-BT*r0);
```

Siehe nächstes Blatt!

```

%% Monte Carlo approach
% BM
BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
CIR = [r0*ones(1,Nplot);zeros(M,Nplot)];
% Euler Maruyama Scheme
for i =1:M
    CIR(i+1,:)=CIR(i,:)+(a-b*CIR(i,:))*dt+ sigma.*sqrt(CIR(i,:)).*(BM(i+1,:)-BM(i,:));
end

mean(CIR(end,:))
var(CIR(end,:))

integral = zeros(1,Nsimu);
intgrid= 0:dt:T;
for j= 1:Nsimu
    % use trapezoidal rule
    integral(j)= trapz(intgrid,CIR(:,j));
end
% take the expectation
valuemc= mean(exp(-integral));
toc
```

6. Matlab File

```
function valuemc=bondDothan
% In this exercise we compute the bond price at time 0 in a Dothan
% shortrate model dr_t = beta r_t dt + sigma r_t d W_t
tic
%% parameter input
% horizon
T=10;
% sample size
Nsimu=10^5;
Nplot=Nsimu;
% grid points
M=10^3;
% volatility
sigma=0.8355;
beta=-0.4454;
```

Bitte wenden!

```
% initial value = r_0
r0=0.08;
% time step
dt= T/M;

%% No closed form solution available

%% Monte Carlo approach
% BM
BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
Dothan = [r0*ones(1,Nplot);zeros(M,Nplot)];
% Euler Maruyama Scheme
for i =1:M
    Dothan(i+1,:)=Dothan(i,:)+ beta*Dothan(i,:)*dt+sigma.*Dothan(i,:).*(BM(i+1,:)-BM(i,:));
end

mean(Dothan(end,:))
var(Dothan(end,:))

integral = zeros(1,Nsimu);
intgrid= 0:dt:T;
for j= 1:Nsimu
    % use trapezoidal rule
    integral(j)= trapz(intgrid,Dothan(:,j));
end
% take the expectation
valuemc= mean(exp(-integral));
toc
```