Exercise Sheet 1

Exercise 1

Let G_1 and G_2 be topological groups. Show that a homomorphism $h : G_1 \to G_2$ is continuous if and only if it is continuous at the identity $e \in G_1$.

NOTE: This exercise exhibits the sharp contrast between continuous homomorphisms from G_1 to G_2 and arbitrary continuous functions from G_1 to G_2.

Exercise 2

(a) Let Λ be a closed subgroup of $(\mathbb{R}, +)$. Show that either

  (i) $\Lambda = \{0\}$,

  (ii) $\Lambda = \alpha\,\mathbb{Z}$ for some $\alpha \in \mathbb{R}_{>0}$, or

  (iii) $\Lambda = \mathbb{R}$.

(b) How many subgroups of $(\mathbb{R}, +)$ are there?

NOTE: There is a wealth of both *closed subsets* (think Cantor set) and *(arbitrary) subgroups* of $\mathbb{R}$ (part (b)). However, (a) shows that *closed subgroups* of $\mathbb{R}$ are quite regular.

Exercise 3

Let X be a compact Hausdorff topological space. Show that $\mathrm{Homeo}(X)$ is a topological group when equipped with the compact-open topology.

NOTE: The result of this exercise may be seen as another argument for why the compact-open topology is a “good” choice.

Exercise 4

Show that $\mathrm{Homeo}(\mathcal{S}^1)$ with the compact-open topology is not locally compact.