Exercise Sheet 3

Exercise 1

(i) Let G be a topological group and let H be a subgroup of G. Equip G/H with the quotient topology. Show that G is connected if both H and G/H are connected.

(ii) Show that $\mathrm{SO}(n)$ is connected for all $n \in \mathbb{N}$.

A similar argument works for $\mathrm{SU}(n)$.

Exercise 2

Let $G = \mathrm{SL}(2,\mathbb{R})$ act on $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$ via fractional linear transformations. Show that this action is transitive, $\mathrm{Stab}_G(i) = \mathrm{SO}(2,\mathbb{R}) =: H$ and that the induced isomorphism of G-spaces $G/H \cong \mathbb{H}$ is a homeomorphism.

Exercise 3

Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}\mathbb{P}^1 = \{V : V \text{ is a 1-dimensional subspace of } \mathbb{R}^2\}$ given by $g.V := gV$ ($g \in \mathrm{SL}(2,\mathbb{R})$, $V \in \mathbb{R}\mathbb{P}^1$). Show directly or by means of Weil's Theorem (Theorem 1.11 in class) that there is no non-trivial Radon measure on $\mathbb{R}\mathbb{P}^1$ that is invariant under the above action of $\mathrm{SL}(2,\mathbb{R})$.

Exercise 4

Let G be a *compact* Hausdorff group with left Haar measure μ_G and let $H < G$ be a closed subgroup of G with left Haar measure μ_H. Show that the proof of Weil's theorem produces the following G-invariant Radon measure $\mu_{G/H}$ on G/H: For all Borel sets $E \subseteq G/H$,

$$\mu_{G/H}(E) = \frac{\mu_G(\pi^{-1}(E))}{\mu_H(H)}, \text{ in particular } \mu_{G/H}(G/H) = \frac{\mu_G(G)}{\mu_H(H)}.$$