ETH Zürich — D-MATH — Introduction to Lie Groups
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Solutions Exercise Sheet 1

Exercise 1

Let G_1 and G_2 be topological groups. Show that a homomorphism $h : G_1 \to G_2$ is continuous if and only if it is continuous at the identity $e \in G_1$.

SOLUTION:

One direction is trivial, namely, if h is continuous then it's obviously continuous at e.

Now for the reverse direction. Assume h is continuous at e. Choose some open set $U \subset G_2$. We want to show that its preimage under h is open also. If $h^{-1}(U) = \emptyset$ we are done. If not then choose some $f(x) \in U$ with $x \in G_1$. We have that:

$$h^{-1}(U) = xh^{-1}(U_e)$$

where $U_e = h(x^{-1})U$. To see this, we observe the following sequence of equivalences: $m \in h^{-1}(U) <=> h(m) \in U <=> h(x^{-1}m) = h(x^{-1})h(m) \in h(x^{-1})U = U_e <=> x^{-1}m \in h^{-1}(U_e) <=> m \in xh^{-1}(U_e)$. Hence $h^{-1}(U) = xh^{-1}(U_e)$. Since translation functions on topological groups are homeomorphisms we have that U_e is open in G_2 and contains $h(e) = e_{G_2}$, hence $h^{-1}(U_e)$ is open in G_1 by hypothesis. Thus $xh^{-1}(U_e) = h^{-1}(U)$ is open and we are done.

Exercise 2

(a) Let Λ be a closed subgroup of $(\mathbb{R}, +)$. Show that either

 (i) $\Lambda = \{0\}$,

 (ii) $\Lambda = \alpha\,\mathbb{Z}$ for some $\alpha \in \mathbb{R}_{>0}$, or

 (iii) $\Lambda = \mathbb{R}$.

SOLUTION:

If $\Lambda = 0$ there is nothing to show. Otherwise, consider $\Lambda_+ = \Lambda \cap \mathbb{R}_{>0}$ and $\Lambda_- = \Lambda \cap \mathbb{R}_{<0} = -\Lambda_+$. Let $\alpha = inf(\Lambda_+)$. Since Λ is closed, $\alpha \in \Lambda$.

If $\alpha > 0$, we claim $\Lambda = \alpha\,\mathbb{Z}$. Obviously it's enough to show $\Lambda_+ = \alpha\,\mathbb{Z}_{>0}$. Pick $x > 0, x \in \Lambda$. By construction $x \geq \alpha$. Since $n\alpha$ goes to ∞ as n goes to ∞, we get that there exists a maximal $m \in \mathbb{Z}_{>0}$ such that $m\alpha \leq x$ and $(m+1)\alpha > x$. But since $\alpha \in \Lambda$, we get that $m\alpha \in \Lambda$ and so $x - m\alpha \in \Lambda$. Since $0 \leq x - m\alpha < \alpha$, we conclude by minimality of α that $0 = x - m\alpha$ (otherwise it would be a positive element of Λ strictly smaller than α). Hence, $\Lambda_+ \subseteq \alpha\,\mathbb{Z}_{>0}$ and obviously $\alpha\,\mathbb{Z}_{>0} \subseteq \Lambda_+$. Thus, conclusion follows.

If $\alpha = 0$, we claim $\Lambda = \mathbb{R}$. Obviously it's enough to show $\Lambda_+ = \mathbb{R}_{>0}$. Pick any $x > 0, x \in \mathbb{R}$. Pick any $\varepsilon > 0$. Set $\varepsilon_x = min(\varepsilon, x) > 0$. Since $\alpha = 0$ there exists $y \in \Lambda$ with $0 < y < \varepsilon_x$. As before, choose m the maximal natural number for which $my \leq x < (m+1)y$. Set $y_1 = my$. As before $y_1 \in \Lambda$ and $0 \leq x - y_1 < y < \varepsilon_x \leq \varepsilon$. So for every $\varepsilon > 0$ we can find a point $y_1 \in \Lambda$ at most ε away from x. But this means that x is a limit point of Λ and since Λ is closed, the conclusion follows.

(b) How many subgroups of $(\mathbb{R},+)$ are there?

SOLUTION:

Let M be the set of all subgroups of $\mathbb{R}$. $\mathbb{R}$ is a $\mathbb{Q}$-vector space in a natural way. Set B to be a $\mathbb{Q}$-basis of $\mathbb{R}$. Define the function $s : 2^B \longrightarrow M$ by mapping $S \mapsto \mathbf{span}(S)$. Notice that this is well defined since every $\mathbb{Q}$-subspace of $\mathbb{R}$ is also a subgroup. Moreover, this function is injective since B is a basis. Hence $|2^B| \leq |M|$. Obviously $|M| \leq |2^{\mathbb{R}}|$ since every subgroup is also a subset.

Let's show $|B| = |\,\mathbb{R}\,|$.

For $i \in \mathbb{N}$ define $X_i = \{x \in \mathbb{R} : x = \lambda_1 x_1 + \ldots + \lambda_i x_i$ for some $\lambda_i \in \mathbb{Q}_{\neq 0}$ and $x_i \in B\}$ with $X_0 = 0$. It is easy to see that X_i form a partition of R. Moreover, it is also easy to see that $|X_i| = |\,\mathbb{Q} \times \ldots \times \mathbb{Q} \times B \times \ldots \times B$ where $\mathbb{Q}$ is multiplied with itself i times and the same with B.

We will invoke the following standard lemma in set theory. if X, Y are two sets such that X is countable and Y is infinite then $|X \times Y| = |Y|$ and $|Y \times Y| = |Y|$.

Using the lemma we get that $|X_i| = |B|$. Since R is a countable disjoint union of the X_is we get that $|R| = |\,\mathbb{N} \times B| = |B|$.

We conclude that $|M| = 2^{\mathbb{R}}$

Exercise 3

Let X be a compact Hausdorff topological space. Show that $\mathrm{Homeo}(X)$ is a topological group when equipped with the compact-open topology.

SOLUTION:

Set $G = Homeo(X)$ and let $m : G \times G \longrightarrow G$ be the multiplication (composition) map and $i : G \longrightarrow G$ be the "taking inverse" map. We have to show that these are continuous with respect to the compact-open topology.

Notice that $i^{-1}(V(K,U) = V(K^c, U^c)$ (K compact and U open). Indeed, we have the following sequence of equivalences: $f \in i^{-1}(V(K,U)) <=> i(f) \in V(K,U) <=> f^{-1} \in V(K,U) <=> f^{-1}(K) \subset U <=> K \subset f(U) <=> f(U)^c \subset K^c <=> f(U^c) \subset K^c <=> f \in V(U^c, K^c)$. In this sequence we have used the fact that f is bijective is order to get that complements of images are images of complements. The fact that X is compact and Hausdorff implies that in X compactness is the same as closedness. Hence K^c is open and U^c is compact. Thus indeed i is continuous.

Let's show that m is continuous. It is enough to show that $m^{-1}(V(K,U))$ is open for every compact K and open U. If this preimage is empty then we are done. If it is not, it is enough to show that for every point $y = (f,g)$ in this preimage there exists an open set $V_y \subset m^{-1}(V(K,U))$ with $y \in V_y$. Then it will follow trivially that the preimage is the union of such V_y and thus open.

So pick $y = (f,g) \in m^{-1}(V(K,U))$. We have that $f(g(K)) \subset U$. Hence $g(K) \subset f^{-1}(U)$. Recall that since X is compact and Hausdorff, closedness and compactness are equivalent. In particular $g(K)$ is closed. Since f is a homeomorphism, we have that $f^{-1}(U^c)$ is also closed and also disjoint from $g(K)$. Again, since X is compact and Hausdorff, it is also normal. Hence, we can separate

$g(K)$ and $f^{-1}(U^c)$ with two disjoint open sets U_1, U_2 such that $g(K) \subset U_1$ and $f^{-1}(U^c) \subset U_2$. Set $K_1 = U_2^c$. We have that

$$g(K) \subset U_1 \subset K_1 \subset f^{-1}(U)$$

Notice that K_1 is closed and thus compact. Also notice that by construction $f \in V(K_1, U)$ and $g \in V(K, U_1)$. So $(f, g) \in V(K_1, U) \times V(K, U_1)$. Set $V_{(f,g)} = V(K_1, U) \times V(K, U_1)$ and notice that this set is open. Moreover, if $(f', g') \in V_{(g,h)}$ then $g' \in V(K, U_1)$ and $f' \in V(K_1, U)$. Hence $f'(K_1) \subset U$ and $g'(K) \subset U_1$. This implies that $f'(g'(K)) \subset f'(U_1) \subset f'(K_1) \subset U$. Thus $V_{(f,g)} \subset m^{-1}(V(K, U))$ and we are done.

Exercise 4

Show that Homeo($\mathcal{S}^1$) with the compact-open topology is not locally compact.

SOLUTION:

Set $G = \text{Homeo}(\mathcal{S}^1)$. It is enough to show that $id \in G$ has no compact neighborhood. Assume the contrary. Then there exists K compact and U open with $K \subset U$ such that $V(K, U) \subset \mathcal{K}$ with $\mathcal{K}$ compact. Since $\mathcal{S}^1$ is a metric space, it follows that the compact-open topology on G coincides with the topology given by compact convergence, or (equivalently) the topology induced by the uniform metric on compact subsets of $\mathcal{S}^1$. Since $\mathcal{S}^1$ is compact itself, this gives that the compact topology on G is given by the uniform metric

$$d_u(f, g) = \sup_{x \in \mathcal{S}^1} d(f(x), g(x))$$

where d is the standard metric on $\mathcal{S}^1$.

Because our goal is to show that id has no compact neighborhood, we want construct small linear perturbations of id. For simplicity we identify $\mathcal{S}^1$ with the interval $[0, 1]$ with the identification $0 \sim 1$. Choose a sequence $1 > \varepsilon_n > 0$ such that ε_n converges to 0. Define the following sequence of functions:

$$f_n(x) = \begin{cases} x + 1 - \frac{\varepsilon_n}{2} & \text{for } 0 \le x \le \frac{\varepsilon_n}{2} \\ 2x - \varepsilon_n & \text{for } \frac{\varepsilon_n}{2} \le x \le \varepsilon_n \\ x & \text{for } \varepsilon_n \le x \le 1 - \varepsilon_n \\ \frac{1}{2}(x + 1 - \varepsilon_n) & \text{for } 1 - \varepsilon_n \le x \le 1 \end{cases}$$

It is easy to check that f_n are well defined homeomorphisms of $\mathcal{S}^1$. we distinguish two cases.

(a) $K = \mathcal{S}^1$ in which case $U = K = \mathcal{S}^1$ since $K \subset U \subset \mathcal{S}^1$. But then $V(K, U) = G$ and $\mathcal{K} = G$ we get that G is compact so f_n must containt a converging subsequence with respect to the d_u metric.

(b) $K \neq \mathcal{S}^1$ in which case, since K is closed, there exists a point x_0 and an open interval around x_0 disjoint of K. After translation by x_0 of K, U and of the f_ns we may assume $x_0 = 0$. But notice that the $f_n = id$ on the interval $[\varepsilon_n, 1 - \varepsilon_n]$. So for n big enough, the f_ns land in $V(K, U)$ and thus in $\mathcal{K}$. So again we get that the f_ns must contain a converging subsequence with respect to the d_u norm.

In either case we obtain a converging subsequence of the f_ns. Pick such a subsequence and denote with f its limit. It is obvious that f_n must converge to f pointwise also. But the f_ns converge pointwise to id. Since X is Hausdorff, we get that $f = id$. However notice that

$$d_u(f_n, id) \geq d(f_n(0), 0) = 1 - \frac{\varepsilon}{2} > \frac{1}{2}$$

Hence no subsequence of the f_ns can converge uniformly to id. Thus we reached a contradiction that arose from the fact that we assumed G to be locally compact. The conclusion follows.