Solutions Exercise Sheet 3

Exercise 1

(i) Let G be a topological group and let H be a subgroup of G. Equip G/H with the quotient topology. Show that G is connected if both H and G/H are connected.

SOLUTION:

Suppose G is not connected. Then $G = G_1 \sqcup G_2$ with G_1, G_2 disjoint non-empty open subsets of G. Let $\pi : G \to G/H$ be the standard projection. We easily get $G/H = \pi(G_1) \cup \pi(G_2)$.

Because π is an open map, $\pi(G_1)$ and $\pi(G_2)$ are non-empty open subsets of G/H. Since G/H is connected, we get that $\pi(G_1) \cap \pi(G_2) \neq \emptyset$. But this implies there exists $g_1 \in G_1$ and $g_2 \in G_2$ such that $\pi(g_1) = \pi(g_2)$ or, equivalently, $g_1^{-1}g_2 \in H$.

Consider $X_1 = H \cap (g_1^{-1}G_1)$ and $X_2 = H \cap (g_1^{-1}G_2)$. Because $e_G \in X_1, g_1^{-1}g_2 \in X_2$ and because translations are homeomorphisms, it follows immediately that X_1 and X_2 are non-empty disjoint open sets of H. Moreover, $H = X_1 \sqcup X_2$ hence contradicting the fact that H is connected. Thus G is connected and we are done.

(ii) Show that $\mathrm{SO}(n)$ is connected for all $n \in \mathbb{N}$.

SOLUTION

We will proceed inductively. For $n = 2$ we have that $\mathrm{SO}(2)$ is homeomorphic to $\mathcal{S}^1$ which is obviously connected.

Define the inclusions $i_n : SO(n) \to \mathrm{SO}(n+1)$

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

It's easy to see that $\mathrm{SO}(n)$ is homeomorphic to $i_n(SO(n))$. Define $j_n : \mathrm{SO}(n+1) \to \mathcal{S}^n$

$$A \mapsto \text{last column of } A$$

It's easy to see that the map j_n is continuous and open as a restriction of a projection from $\mathbb{R}^{(n+1)^2}$ to $\mathbb{R}^{n+1}$ (remember that projections are open maps). Moreover j_n is surjective and $j_n(A) = j_n(B)$ if and only if $A^{-1}B = A^TB \in i_n(SO(n))$. Given that the projection $SO(n+1) \to SO(n+1)/i_n(SO(n))$ is an open map also, we get that j_n descends to a continuous bijective open map, hence a homeomorphism $SO(n+1)/i_n(SO(n) \simeq \mathcal{S}^n$.

Since $i_n(SO(n))$ is connected by the induction hypothesis and since $\mathcal{S}^n$ is obviously connected, we get by item (i) that $SO(n+1)$ is connected. Thus the induction step is completed and the conclusion follows.

Exercise 2

Let $G = \mathrm{SL}(2,\mathbb{R})$ act on $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$ via fractional linear transformations. Show that this action is transitive, $\mathrm{Stab}_G(i) = \mathrm{SO}(2,\mathbb{R}) =: H$ and that the induced isomorphism of G-spaces $G/H \cong \mathbb{H}$ is a homeomorphism.

SOLUTION:

Pick $z = x + iy \in \mathbb{H}$. Let's show that z is in i's orbit. We need to find $A \in \mathrm{SL}(2,\mathbb{R})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A.i = z$ or, equivalently

$$\begin{cases} \dfrac{ac+bd}{c^2+d^2} = x \\ \dfrac{1}{c^2+d^2} = y \end{cases}$$

Let's try to find a solution for this system with $c = 0$. Then we can choose $d = \dfrac{1}{\sqrt{y}}$ which then forces $a = \sqrt{y}$ (because $\det(A) = 1$). Finally, the first equation (after all substitutions are made) gives us $\dfrac{b}{\sqrt{y}} = x$, or $b = x\sqrt{y}$. So we found $A = \begin{pmatrix} \sqrt{y} & x\sqrt{y} \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix}$ such that $A.i = z$. Thus the action is transitive.

If $A.i = i$ then

$$\begin{cases} \dfrac{ac+bd}{c^2+d^2} = 0 \\ \dfrac{1}{c^2+d^2} = 1 \end{cases}$$

Hence $c^2 + d^2 = 1$, $ac + bd = 0$ and $ad - bc = 1$. Squaring and adding the last two equations gives us $a^2 + b^2 = 1$. Now

$$AA^t = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means that $A \in H$. Hence $\mathrm{Stab}_G(i) \subseteq H$. The reverse inclusion is immediate. Thus $\mathrm{Stab}_G(i) = H$.

The induced isomorphism is given by $(A \bmod H) \mapsto A.i$. It is easy to see that this is a continuous mapping (just look at preimages of balls in $\mathbb{C}$ and take into account that the topology on $\mathrm{SL}(2,\mathbb{R})$ is the subspace topology from $GL(2,\mathbb{R})$ which is nothing more than the Euclidian topology in $\mathbb{R}^4$). Now we want to check that the inverse is also continuous. The above discussion shows that the inverse of this isomorphism is given by the composition of the map $\phi : \mathbb{H} \to G)$ given by $x+iy \mapsto \begin{pmatrix} \sqrt{y} & x\sqrt{y} \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix}$, with the projection map $G \to G/H$. It is easy to see that ϕ is a continuous map since all the entries of the image matrix are continuous maps. Moreover, we know that this is also true for the projection map. So the inverse is also continuous hence the isomorphism is also a homeomorphism.

Exercise 3

Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}\mathbb{P}^1 = \{V : V \text{ is a 1-dimensional subspace of } \mathbb{R}^2\}$ given by $g.V := gV$ ($g \in \mathrm{SL}(2,\mathbb{R}),\ V \in \mathbb{R}\mathbb{P}^1$). Show directly or by means of Weil's Theorem (Theorem 1.11 in class) that there is no non-trivial Radon measure on $\mathbb{R}\mathbb{P}^1$ that is invariant under the above action of $\mathrm{SL}(2,\mathbb{R})$.

SOLUTION:

Set $G = \mathrm{SL}(2,\mathbb{R})$ and consider the map $\phi : G \to \mathbb{R}\mathbb{P}^1$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathrm{span}\left\{\begin{pmatrix} a \\ c \end{pmatrix}\right\}$$

It is easy to see that ϕ is G-equivariant and surjective. Moreover, if we let H to be the subgroup of G given by

$$H = \{\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R}_{\neq 0}\}$$

Then we notice that $\phi(g_1) = \phi(g_2)$ iff $g_1^{-1}g_2 \in H$. This is because if we let

$$g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then } g_2 = \begin{pmatrix} ka & b' \\ kc & d' \end{pmatrix} \quad \text{for some } k \in \mathbb{R}_{\neq 0}, b' \in \mathbb{R}, d' \in \mathbb{R} \quad \text{and } k(ad' - cb') = 1.$$

But then

$$g_1^{-1}g_2 = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} ka & b' \\ kc & d' \end{pmatrix} = \begin{pmatrix} k & db' - bd' \\ 0 & \frac{1}{k} \end{pmatrix} \in H$$

This means that ϕ descends to a bijective G-equivariant map $\overline{\phi} : G/H \to \mathbb{R}\mathbb{P}^1$ given by $\overline{\phi}(gH) = \phi(g)$. Moreover if $\pi : G \to G/H$ is the canonical projection then we have that $\phi = \overline{\phi} \circ \pi$.

It is easy to check that $\overline{\phi}$ is continuous and that ϕ is an open map (look at preimages of open balls in the standard affine patches of $\mathbb{R}\mathbb{P}^1$ for continuity of ϕ and at images of open balls in $\mathbb{R}^2 \cap \mathrm{SL}(2,\mathbb{R})$ to check that ϕ is an open mapping). We want now to check that $\overline{\phi}$ is an open map. It will then follow that ϕ is a homeomorphism.

Pick $E \subset G/H$ open and set $E' = \pi^{-1}(E)$. Since π is surjective we have that $\pi(E') = E$ and since π is continuous, we get that E' is open in G. But then $\overline{\phi}(E) = \overline{\phi}(\pi(E')) = \phi(E')$ which is open since ϕ is an open map. Thus the G-isomorphism $\overline{\phi}$ is also a homeomorphism.

This implies that we have to show that there is no G-invariant measure on G/H. However, by Weil's theorem this happens iff $\Delta_G(x) = \Delta_H(x)$ for every $x \in H$.

However we have already seen that $\Delta_G(x) = 1$ for all $x \in G$ whereas

$$\Delta_H(\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}) = \frac{1}{a^2}$$

Since the two modular functions don't agree on H we are done.

Exercise 4

Let G be a *compact* Hausdorff group with left Haar measure μ_G and let $H < G$ be a closed subgroup of G with left Haar measure μ_H. Show that the proof of Weil's theorem produces the following G-invariant Radon measure $\mu_{G/H}$ on G/H: For all Borel sets $E \subseteq G/H$,

$$\mu_{G/H}(E) = \frac{\mu_G(\pi^{-1}(E))}{\mu_H(H)}, \text{ in particular } \mu_{G/H}(G/H) = \frac{\mu_G(G)}{\mu_H(H)}.$$

SOLUTION:

Let's try a "sloppy" proof first. We have that

$$\mu_{G/H}(E) = \int_{G/H} char_E(xH) d\mu_{G/H}(xH)$$

where $char_E$ is the characteristic function of E. Recall now that from the proof of Weil's Theorem we get that if T_H is the map from $C_c(G)$ to $C_c(G/H)$ for which $T_H(f)(xH) = \int_H f(xh) d\mu_H(h)$, and $T_H(f) = char_E$ for some $f \in C_c(G)$ then $\int_{G/H} char_E(xH) d\mu_{G/H}(xH) = \int_G f(x) d\mu_G(x)$.

Take $f = \frac{1}{\mu_H(H)} char_{\pi^{-1}(E)}$. Then for $xH \in E$ we have that $xH = xhH$ for all $h \in H$, so $xh \in \pi^{-1}(E)$ for all $h \in H$ and thus $T_H(f)(xH) = \frac{1}{\mu_H(H)} \int_H 1 \, d\mu_H(h) = 1$. Similarly, if $xH \notin E$ then $xhH \notin E$ for all $h \in H$ so $xh \notin \pi^{-1}(E)$ for all $h \in H$ and thus $T_H(f)(xH) = \frac{1}{\mu_H(H)} \int_H 0 \, d\mu_H(h) = 0$. We get that $T_H(f) = char_E$. Thus

$$\mu_{G/H}(E) = \int_{G/H} char_E(xH) d\mu_{G/H}(xH) = \int_G f(x) d\mu_G(x) =$$

$$= \frac{1}{\mu_H(H)} \int_G char_{\pi^{-1}(E)}(x) d\mu_G(x) = \frac{\mu_G(\pi^{-1}(E))}{\mu_H(H)}$$

The problem with this "sloppy" proof is that characteristic functions are not continuous in general. This is indeed a problem because so far we associated Radon measures and Haar measure with functionals on the space of compactly supported continuous functions. The continuity requirement cannot be completely dropped for Radon measures due to certain issues with respect to regularity conditions. It can be shown that Haar measure on the other hand can be associated to functionals on the larger space of integrable functions (not just continuous) in which case the "sloppy" proof above is perfectly valid. However, for consistency purposes, I will now use a different method that avoids this "mistake".

The point is that Haar measures are unique up to constants. So let's check that the measure $\mu'_{G/H}$ defined by

$$\mu'_{G/H}(E) = \frac{\mu_G(\pi^{-1}(E))}{\mu_H(H)}$$

is G-invariant. For this we notice that $\pi^{-1}(x.E) = x\pi^{-1}(E)$ (where $a.(bH)$ should read a acting on $bH \in G/H$) for all $x \in G$ (this is a trivial consequence of the fact that π is G-equivariant). This means that

$$\mu'_{G/H}(x.E) = \frac{\mu_G(x\pi^{-1}(E))}{\mu_H(H)} = \frac{\mu_G(\pi^{-1}(E))}{\mu_H(H)}$$

where the last equality uses G-invariance of μ_G. This means that $\mu'_{G/H}$ is Haar. But then by uniqueness of Haar measures we get that there exist a constant $K > 0$ such that $\mu_{G/H} = K\mu'_{G/H}$. So now we only have to show $K = 1$. But for this we notice that

$$\mu'_{G/H}(G/H) = \frac{\mu_G(G)}{\mu_H(H)} \quad \text{since } \pi^{-1}(G/H) = G.$$

However now we can use our "sloppy" proof since $char_{G/H}$ is continuous on G/H and since $char_G$ is continuous on G. This gives us that

$$\mu_{G/H}(G/H) = \frac{\mu_G(G)}{\mu_H(H)}$$

Combining the last two equalities gives us $K = 1$ and thus we are done.