

Mathematical Foundations For Finance

Exercise Sheet 1

This sheet contains material which is **fundamental** for this course and assumed to be **known**.

Exercise 1-1. Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ be a finite set and $X : \Omega \rightarrow \mathbb{R}$ a mapping which takes the values $+5, 0$ and -5 . You can think of X as a stock price change over one time period.

- What is the σ -field $\sigma(X)$ generated by X ?
- Show that $|X|$ is measurable with respect to $\sigma(X^2)$.
- Let $Y : \Omega \rightarrow \mathbb{R}$ be another function. If $\sigma(Y) = 2^\Omega$, what can you say about Y ?

Exercise 1-2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space with $\Omega := \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} := 2^\Omega$ and \mathbb{P} defined by $\mathbb{P}[\{\omega_1\}] := 0.5$ and $\mathbb{P}[\{\omega_3\}] := 0.2$. Let $X : \Omega \rightarrow \mathbb{R}$ be the random variable defined by $X(\omega_1) := 12$, $X(\omega_2) := -4$ and $X(\omega_3) := -8$ and $\mathcal{G} := \{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}$ a sub- σ -field of \mathcal{F} .

- What is $\mathbb{P}[\{\omega_2\}]$?
- Think of X as the price change over one week of a Facebook share. Give a verbal description of the set $A := \{\omega_2, \omega_3\}$ in terms of X .
- Calculate the conditional expectation $\mathbb{E}[X | \mathcal{G}]$.
- Find all probability measures \mathbb{Q} that are equivalent to \mathbb{P} on \mathcal{F} and satisfy

$$\mathbb{E}_{\mathbb{Q}}[X] = 4,$$

where $\mathbb{E}_{\mathbb{Q}}[\cdot]$ denotes the expectation under \mathbb{Q} .

Exercise 1-3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \rightarrow \mathbb{R}$ a random variable with $X \geq 0$ \mathbb{P} -a.s.. Prove that $\mathbb{E}[X] = 0$ implies that $X = 0$ \mathbb{P} -a.s..

Exercise 1-4. Exercise 1-2 refers to the price change over one week of a Facebook share. We want to check if the values given for this weekly price change are realistic. We can answer this question with the software R.

- Use the function `getSymbols` from the R package `quantmod`, to extract the prices of a Facebook share from Yahoo Finance.
- Find the weekly price changes of a Facebook share over these three weeks : from the 18.08.14 to the 22.08.14, from the 25.08.14 to the 29.08.14, and from the 01.09.14 to the 05.09.14. Are these price changes consistent with the values from 1-2 ?

Remark. You might need to install the package `quantmod`. Once it is done don't forget to load it with the command `library(quantmod)`.