ETH Zürich HS 2014
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 10

Please hand in by Wednesday, 26/11/2013, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 10-1. A *Poisson process with parameter* $\lambda > 0$ *with respect to a probability measure* $\mathbb{P}$ *and a filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a (real-valued) stochastic process $N = (N_t)_{t\geq 0}$ which is adapted to $\mathbb{F}$, starts at 0 (i.e. $N_0 = 0$ $\mathbb{P}$-a.s. .) and satisfies the following two properties:

(PP1) For $0 \leq s < t$, the *increment* $N_t - N_s$ is independent (under $\mathbb{P}$) of $\mathcal{F}_s$ and is (under $\mathbb{P}$) *Poisson-distributed with parameter* $\lambda(t-s)$, i.e.

$$\mathbb{P}[N_t - N_s = k] = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for $\mathbb{P}$-almost all ω, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant and $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Poisson processes form the cornerstone of *jump processes*, which are of importance in advanced financial modelling. Show that the following processes are $(\mathbb{P}, \mathbb{F})$-martingales:

(a) $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. This is also called a *compensated Poisson process.*

Hint: If $X \sim Poi(\lambda)$, then $\mathbb{E}[X] = \lambda$.

(b) $(\widetilde{N}_t)^2 - N_t$, $t \geq 0$, and $(\widetilde{N}_t)^2 - \lambda t$, $t \geq 0$.

Hint: If $X \sim Poi(\lambda)$, then $\mathrm{Var}[X] = \lambda$.

(c) $S_t := e^{\log(1+\sigma)N_t - \lambda\sigma t}$, $t \geq 0$, where $\sigma > -1$. This is also called a *geometric Poisson process.*

Exercise 10-2. The aim of this exercise is to compute hedging strategies in the Black-Scholes model. We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there exists a standard Brownian motion W. This Brownian motion generates a filtration, that we augment with the $\mathbb{P}$ null-sets. We call $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$, the resulting filtration. The discounted price process (we assume r=0 for simplicity) is given by:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right).$$

Let $K \in \mathbb{R}$. Find the hedging strategies of the contingent claims $h(S_T)$ for the payoff functions:

(a) $h(y) = y^4 - 6y^2 + 3$, (5^{th} Hermite polynomial).

(b) $h(y) = \mathbb{1}_{\{y \geqslant K\}}$, digital option.

(c) $h(y) = (K - y)^+$, European put option.

Exercise 10-3. Let $W = (W_t)_{t\geq 0}$ be a BM.

Please see next page!

(a) Show that for $\alpha, \sigma \in \mathbb{R}$ the process X defined by

$$X_t := e^{-\alpha t}\left(X_0 + \sigma \int_0^t e^{\alpha s}\, \mathrm{d}W_s\right) \qquad t \geq 0\,,$$

satisfies the *stochastic differential equation* (SDE)

$$X_t - X_0 = -\alpha \int_0^t X_s\, \mathrm{d}s + \sigma W_t \qquad t \geq 0\,.$$

(b) Show that for $a \in \mathbb{R}$ the processes $Z^{(1)}$ and $Z^{(1)}$ defined by

$$Z_t^{(1)} := \cos(aW_t) \qquad \text{and} \qquad Z_t^{(2)} := \sin(aW_t) \quad t \geq 0,$$

satisfy the system of SDEs

$$\begin{cases} Z_t^{(1)} = 1 - \frac{a^2}{2}\int_0^t Z_s^{(1)}\, \mathrm{d}s - a\int_0^t Z_s^{(2)}\, \mathrm{d}W_s\,, \\ Z_t^{(2)} = -\frac{a^2}{2}\int_0^t Z_s^{(2)}\, \mathrm{d}s + a\int_0^t Z_s^{(1)}\, \mathrm{d}W_s \end{cases} \qquad t \geq 0\,.$$

(c) Let $S = (S_t)_{t\geq 0}$ be given by

$$S_t = S_0 + \int_0^t \mu_u S_u\, \mathrm{d}u + \int_0^t \sigma_u S_u\, \mathrm{d}W_u \quad t \geq 0\,,$$

where $\mu = (\mu_t)_{t\geq 0}$ and $\sigma = (\sigma_t)_{t\geq 0}$ are predictable with $\int_0^T |\mu_s| + |\sigma_s|^2 ds < \infty$ $\mathbb{P}$-a.s. for all $T > 0$. For fixed $T > 0$, show that

$$\int_0^T \sigma_s^2\, \mathrm{d}s = -2\log\frac{S_T}{S_0} + \int_0^T \frac{2}{S_u}\, \mathrm{d}S_u \qquad \mathbb{P}\text{-a.s.}\,.$$

In terms of mathematical finance, this means that the integrated local variance of S has the same price as $-2\log$ contracts on S.

Hint. Argue that $S_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.