ETH Zürich HS 14
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 11

Please hand in by Wednesday, 3/12/2013, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 11-1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$. We consider a Poisson process $(N_t)_{t \geqslant 0}$ with intensity parameter $\lambda > 0$.

(a) Compute the characteristic function of N_t.

(b) Show that the sequence of random variables $\left(\frac{1}{\sqrt{c}}(N_{ct} - \lambda ct)\right)_{c \in \mathbb{N} \setminus \{0\}}$ converges in law to a normal random variable X_t with mean 0 and variance λt.

(c) Assume that the sequence of processes $\left(\left(\frac{1}{\sqrt{c}}(N_{ct} - \lambda ct)\right)_{t \geqslant 0}\right)_{c \in \mathbb{N} \setminus \{0\}}$ converges almost surely to a process $(X_t)_{t \geqslant 0}$. Prove that the process X has independent increments.

Exercise 11-2. Let $W = (W_t)_{t \geq 0}$ be a Brownian motion with respect to a probability measure $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Starting from Itô's formula, decide for each of the following processes whether or not they are $(\mathbb{P}, \mathbb{F})$-martingales:

(a) $X_t^{(1)} := W_t^p - ptW_t$, $t \geq 0$, and $p \geq 2$.

(b) $X_t^{(2)} := \exp\left(\frac{1}{2}\alpha^2 t\right) \cos\left(\alpha\left(W_t - \beta\right)\right)$, $t \geq 0$, where $\alpha \neq 0$ and $\beta \in \mathbb{R}$.

(c) $X_t^{(3)} := \sin W_t - \cos W_t$, $t \geq 0$.

Exercise 11-3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, W a Brownian motion and $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ the augmented filtration generated by W. We consider the Black-Scholes model, with $r = 0$. The stock price process is given by:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right). \qquad (1)$$

The purpose of this exercise is to find the probability change that makes S a martingale.

(a) Compute $\mathbb{E}\left[S_t \mid \mathcal{F}_s\right]$ for $0 \leqslant s < t$.

(b) We define the process Z by $Z_t = \exp\left(-\frac{\mu}{\sigma}W_t - \frac{\mu^2}{2\sigma^2}t\right)$. Prove that this process is a positive martingale under $\mathbb{P}$, and that $\mathbb{E}\left[Z_t\right] = 1$ for all $t \in [0, T]$.

(c) Define $\mathbb{Q}$ as the probability measure whose density process with respect to $\mathbb{P}$ is Z. Prove that the process $\widetilde{W}$ defined by $\widetilde{W}_t = W_t + \frac{\mu}{\sigma}t$ is a $\mathbb{Q}$-Brownian motion. Prove then that S is a martingale under $\mathbb{Q}$.