ETH Zürich HS 14
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 12

Please hand in by Wednesday, 10/12/2013, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 12-1. The aim of this exercise is to use a relation between American and European contingent claims to compute the price of an American option.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space, on which exists a Brownian motion W. We consider a Bachelier market model with two assets :

$$S_t^0 \equiv 1,$$
$$S_t^1 = S_0^1 + \sigma W_t,$$

and we define the process $X^{t,x}$ by $X_s^{t,x} = x + \sigma (W_s - W_t)$ for $s \in [t,T]$. It is the price process of the risky asset given that its price at t is x.

Let g be a measurable function such that $g\left(S_t^1\right)$ is in L^1 for all $t \in [0,T]$. We consider the European contingent claim with payoff $g\left(S_T\right)$ at time T. Define

$$u\left(x,t\right) = \mathbb{E}\left[g\left(x + \sigma\left(W_T - W_t\right)\right)\right].$$

Assume that u is a continuous function of its arguments.

The price process V of the European option with terminal payoff $g(S_T^1)$ is then given by (see Exercise 9-3):

$$\widetilde{V}_t = u\left(S_0^1 + \sigma W_t, t\right).$$

Define

$$v_g(x) = \inf_{s\in[0,T]} u\left(x,s\right).$$

We assume that the infimum is attained at a point that we call $t(x)$, and that the function $x \mapsto t(x)$ is continuous.

We consider now the American option with payoff process U, defined as $U_t = v_g\left(S_t^1\right)$. Let $C_t := \{x \in \mathbb{R} \mid t(x) \geqslant t\}$ be the continuation region of the option at time t. For $x \in C_t$ the price of the American option at time t is given by :

$$V_t = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[v_g\left(X_\tau^{t,S_0^1+\sigma W_t}\right) \mid \mathcal{F}_t\right],$$

where $\mathcal{T}_{t,T}$ is the set of stopping times that takes value in $[t,T]$.

(a) Prove that the process $Z = \left(u\left(X_s^{t,x}, s\right)\right)_{t\leqslant s\leqslant T}$ is a martingale under $\mathbb{P}$.

(b) Prove that: $\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[v_g\left(X_\tau^{t,x}\right) \mid \mathcal{F}_t\right] \leqslant u(x,t)$

Hint. Use part a.

(c) Prove that for $x \in C_t$, the following holds: $\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[v_g\left(X_\tau^{t,x}\right) \mid \mathcal{F}_t\right] \geqslant u(t,x)$.

Hint. Define the stopping time $\widetilde{\tau} = \inf\{s \in [t,T] \mid s = t(X_s^{x,t})\} \wedge T$. You don't have to prove that it is a stopping time.

(d) Let $g : x \mapsto x^4 - 10x^2 + 5$. Find v_g, and compute the price of the American contingent claim with payoff process $v_g\left(S_t^1\right)$.

Exercise 12-2. Let $W = (W_t)_{t \geq 0} = (W_t^1, W_t^2, \dots, W_t^m)_{t \geq 0}$ be an $\mathbb{R}^m$-valued Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Show that for $k \neq \ell$ the process $W^k W^\ell$ is a martingale.

(b) Conclude that $[W^k, W^\ell]_t = \delta_{k\ell} t$, for $t \geq 0$, and $k, \ell \in \{1, \dots, m\}$.

Exercise 12-3. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]})$ be a filtered probability space and consider two *independent* Brownian motions $W^1 = (W_t^1)_{t \in [0,T]}$ and $W^2 = (W_t^2)_{t \in [0,T]}$. Let $\widetilde{S}^1 = (\widetilde{S}_t^1)_{t \in [0,T]}$ and $\widetilde{S}^2 = (\widetilde{S}_t^2)_{t \in [0,T]}$ be two *undiscounted* stock price processes with the following *dymanics*

$$\mathrm{d}\widetilde{S}_t^1 = \widetilde{S}_t^1 (\mu_1 \,\mathrm{d}t + \sigma_1 \,\mathrm{d}B_t^1), \qquad \widetilde{S}_0^1 > 0,$$

$$\mathrm{d}\widetilde{S}_t^2 = \widetilde{S}_t^2 (\mu_2 \,\mathrm{d}t + \sigma_2 \,\mathrm{d}B_t^2), \qquad \widetilde{S}_0^2 > 0,$$

where $B^1 = W^1$, $B^2 = \alpha W^1 + \sqrt{1 - \alpha^2} W^2$, for some $\alpha \in [0, 1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

(a) Apply Itô's formula to $X^1 := \frac{\widetilde{S}^2}{\widetilde{S}^1}$ and $X^2 := \frac{\widetilde{S}^1}{\widetilde{S}^2}$.

Remark. Since $\widetilde{S}^1$ and $\widetilde{S}^2$ have continuous trajectories and satisfy $\widetilde{S}_t^1, \widetilde{S}_t^2 > 0$ for all $t \in [0, T]$ $\mathbb{P}$-a.s., we can choose each of them as *numéraire.*

(b) For $\beta_1, \beta_2 \in \mathbb{R}$, define the continuous $(\mathbb{P}, \mathbb{F})$-martingale $L^{(\beta_1, \beta_2)} := \beta_1 W^1 + \beta_2 W^2$.

We define the stochastic exponential $\mathcal{E}(X)$ as follows:

$$\mathcal{E}(X)_t := \exp\left(X_t - X_0 - \frac{1}{2} \langle X \rangle_t \right)$$

Show that for all $\beta_1, \beta_2 \in \mathbb{R}$ the stochastic exponential $Z^{(\beta_1, \beta_2)} := \mathcal{E}(L^{(\beta_1, \beta_2)})$ is a $(\mathbb{P}, \mathbb{F})$-martingale on $[0, T]$.

Hint. You can use the following facts: a continuous process integrated with respect to a continuous martingale is a local martingale, a nonnegative local martingale is a supermartingale, and a supermartingale with constant expectation is a true martingale.

The two following questions can be left out. They are a bit more involved mathematically, but are a nice exercise for the use of Girsanov's theorem.

*(c)*** For $\beta_1, \beta_2 \in \mathbb{R}$, define by $\mathrm{d}\mathbb{Q}^{(\beta_1, \beta_2)} = Z_T^{(\beta_1, \beta_2)} \,\mathrm{d}\mathbb{P}$ a probability measure $\mathbb{Q}^{(\beta_1, \beta_2)}$ which is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$. Fix $\beta_1, \beta_2 \in \mathbb{R}$. Using Girsanov's theorem (Theorem 6.2.3 in the lecture notes), show that the two processes $\widetilde{W}_t^1 := W_t^1 - \beta_1 t$ and $\widetilde{W}_t^2 := W_t^2 - \beta_2 t$, $t \in [0, T]$, are local $(\mathbb{Q}^{(\beta_1, \beta_2)}, \mathbb{F})$-martingales. Conclude that

$$\widetilde{B}^1 := \widetilde{W}^1 \quad \text{and} \quad \widetilde{B}_t^2 := B_t^2 - (\alpha \beta_1 + \sqrt{1 - \alpha^2} \beta_2) t, \quad t \in [0, T],$$

are local $(\mathbb{Q}^{(\beta_1, \beta_2)}, \mathbb{F})$-martingales as well.

Remark. One can show that $\widetilde{W}^1$ and $\widetilde{W}^2$ are *independent* Brownian motions under $\mathbb{Q}^{(\beta_1, \beta_2)}$ and correspondingly that $\widetilde{B}^1$ and $\widetilde{B}^2$ are *correlated* Brownian motions under $\mathbb{Q}^{(\beta_1, \beta_2)}$.

*(d)*** What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the proces X^1 respectively X^2 a $(\mathbb{Q}^{(\beta_1, \beta_2)}, \mathbb{F})$-martingale?

Exercise 12-4. Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t \in [0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the filtration generated by W

and augmented by the $\mathbb{P}$-nullsets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$\frac{\mathrm{d}\widetilde{S}^0_t}{\widetilde{S}^0_t} = r\,\mathrm{d}t \quad \text{and} \quad \frac{\mathrm{d}\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$. Denote by $\mathbb{Q}^*$ the unique equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$ on $\mathcal{F}_T$.

(a) Hedge the *square option*, i.e., find (V_0, ϑ) such that

$$V_0 + \int_0^T \vartheta_u\,\mathrm{d}S^1_u = \frac{(\widetilde{S}^1_T)^2}{\widetilde{S}^0_T}.$$

Hint. Look for a representation result under $\mathbb{Q}^*$, not under $\mathbb{P}$.
The formula $\mathbb{E}\left[e^{uX}\right] = e^{\frac{1}{2}u^2\sigma^2}$ for $X \sim \mathcal{N}(0,\sigma^2)$ and $u \in \mathbb{R}$ may be useful.

(b) Hedge the *inverted option*, i.e., find $(\overline{V}_0, \overline{\vartheta})$ such that

$$\overline{V}_0 + \int_0^T \overline{\vartheta}_u\,\mathrm{d}S^1_u = \frac{1}{\widetilde{S}^0_T \widetilde{S}^1_T}.$$