ETH Zürich HS 14
D-MATH
Prof. J. Teichmann

Coordinator
Thomas Cayé

Mathematical Foundations For Finance

Exercise Sheet 13

Please hand in by Wednesday, 17/12/2013, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 13-1. Let $T > 0$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the $\mathbb{P}$-null sets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model with *stochastic bank account*. In this model, the undiscounted bank account price process $\widetilde{S}^0$ and the undiscounted stock price process $\widetilde{S}^1$ are given by

$$\mathrm{d}\widetilde{S}^0_t = \widetilde{S}^0_t(r\,\mathrm{d}t + \gamma\,\mathrm{d}W_t), \qquad \widetilde{S}^0_0 = 1,$$

$$\mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t\left(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right), \quad \widetilde{S}^1_0 = s > 0,$$

where $r, \mu \in \mathbb{R}$, $\gamma \ge 0$ and $\sigma > 0$. Note that for $\gamma = 0$, this corresponds to the standard Black–Scholes model. Denote by $S^1 := \widetilde{S}^1/\widetilde{S}^0$ the discounted stock price process.

(a) First, assume that $\gamma \neq \sigma$. Find a measure $Q^* \approx P$ on $\mathcal{F}_T$ such that S^1 is a Q^*-martingale and show that, under Q^*, S^1 satisfies the SDE

$$\mathrm{d}S^1_t = S^1_t(\sigma - \gamma)\,\mathrm{d}W^*_t, \quad S^1_0 = s,$$

where $W^* = (W^*_t)_{t\in[0,T]}$ is a Q^*-Brownian motion.

Remark. One can show that Q^* is the *unique* equivalent martingale measure for S^1.

(b) Next, assume that $\gamma = \sigma > 0$ and $\mu > r$. Show that the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits arbitrage by explicitly constructing an arbitrage opportunity, i.e., an admissible self-financing strategy $\phi = (\eta, \theta)$ with $\phi_0 = (0, 0)$ (so that $V_0(\phi) = 0$) such that

$$V_T(\phi) = \int_0^T \theta_u\,\mathrm{d}S^1_u \ge 0 \;\; \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[V_T(\phi) > 0] > 0.$$

Hint. You can choose a buy-and-hold strategy in the stock, i.e., $\theta = c\mathbb{1}_{]\!]0,T]\!]}$, where $c \in \mathbb{R}$ is a constant. Moreover, don't forget to specify η.

(c) Finally, assume that $\gamma = 0$, $\sigma > 0$ and $r, \mu \in \mathbb{R}$, i.e., we are in the setting of the standard Black–Scholes model. A *power option* with power 4 and strike $\widetilde{K} \ge 0$ on $\widetilde{S}^1$ is a contingent claim whose undiscounted payoff a time T is given by

$$\widetilde{H}^{\text{pow}} := \left(\left(\widetilde{S}^1_T\right)^4 - \widetilde{K}\right)^+.$$

Show that

$$\mathbb{E}_{Q^*}\left[\frac{\widetilde{H}^{\text{pow}}}{\widetilde{S}^0_T}\right] \ge \left(s^4 \exp\left((6\sigma^2 + 3r)T\right) - \widetilde{K}\exp(-rT)\right)^+,$$

where Q^* is the unique equivalent martingale measure for S^1.

Hint. In a first step, assume that $\widetilde{K} = 0$.

Exercise 13-2. Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W

and augmented by the $\mathbb{P}$-nullsets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model, where the (undiscounted) bank account price process $\widetilde{S}^0$ and the (undiscounted) stock price process $\widetilde{S}^1$ are given by $\widetilde{S}^0_t = e^{rt}$ and $\widetilde{S}^1_t = e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$, $0 \le t \le T$, for some fixed $r, \mu \in \mathbb{R}$ and $\sigma > 0$. Denote by $\mathbb{Q}$ the unique equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$ on $\mathcal{F}_T$. Moreover, recall the discounted *Black-Scholes formula*, which computes the (discounted) price process $V^{\mathrm{Call},K}$ of a (discounted) European call option with maturity $T > 0$ and (discounted) strike $K > 0$, i.e.,

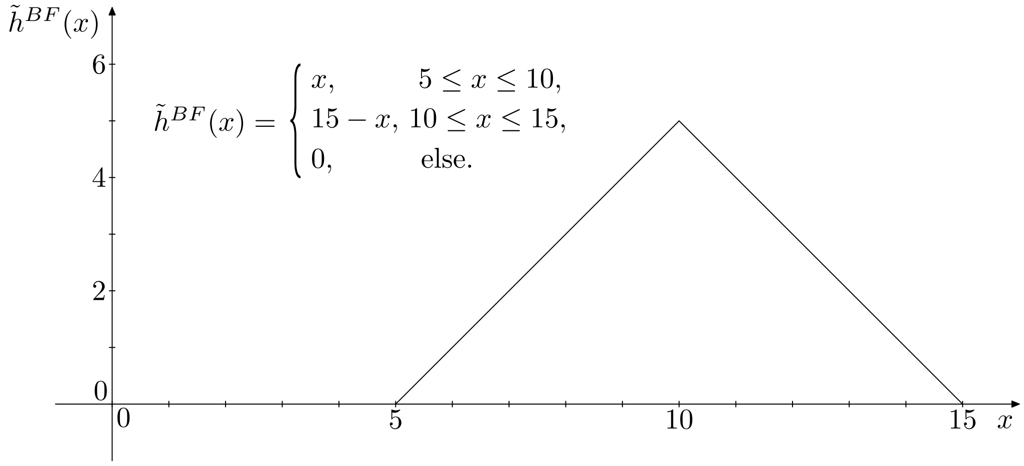
$$V_t^{\mathrm{Call},K} = S_t^1 \Phi(d_1(K,t)) - K\Phi(d_2(K,t)),$$

where

$$d_{1,2}(K,t) = \frac{\log\left(S_t^1/K\right) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and Φ denotes the cumulative distribution function of a standard normal random variable.

(a) Consider a European *butterfly option* with the following payoff structure



This means that the graph of $\widetilde{h^{BF}}$ is depicted, where $\widetilde{H^{BF}} = \widetilde{h^{BF}}(\widetilde{S}^1_T)$ denotes the corresponding payoff of a *butterfly option*.

(i) Show that the above *butterfly option* can be represented by a linear combination of European call options, i.e., find $\alpha, \beta, \gamma, \widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3 \in \mathbb{R}$ such that

$$\widetilde{h}^{BF}(s) = \alpha(s - \widetilde{K}_1)^+ + \beta(s - \widetilde{K}_2)^+ + \gamma(s - \widetilde{K}_3)^+.$$

(ii) Hedge the discounted *butterfly option*, i.e., find $(V_0^{BF}, \vartheta^{BF})$ such that

$$H^{BF} := \frac{\widetilde{H}^{BF}}{\widetilde{S}^0_T} = V_0^{BF} + \int_0^T \vartheta_t^{BF} S_t^1, \quad \text{a.s.}$$

Remark: If you cannot solve part (i), you may try to solve part (ii) with general $\alpha, \beta, \gamma, \widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3$.

(b) Compute the price at time 0 of the discounted option

$$H^{\log} := (\log S_T^1)^4.$$

Hint: If $X \sim \mathcal{N}(0,1)$, then $E[X^4] = 3$. **(Please turn page)**

Please see next sheet!

(c) Let $f : (0, \infty) \to (0, \infty)$ be a concave function. Consider the discounted option

$$\overline{H}^f := S^1_T f\left(\frac{1}{S^1_T}\right).$$

Show that the arbitrage-free price $V^{\overline{H}^f}$ of the option satisfies

$$V_t^{\overline{H}^f} \leq S^1_t f\left(\frac{1}{S^1_t}\right), \ \forall t \in [0, T] \quad \text{a.s.}$$

Exercise 13-3. Let $T > 0$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the $\mathbb{P}$-null sets in $\sigma(W_s; 0 \leq s \leq T)$.

First, consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0$ and the undiscounted stock price process $\widetilde{S}^1$ are given by $\widetilde{S}^0_t := e^{rt}$ and $\widetilde{S}^1_t := e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$, $0 \leq t \leq T$, $r, \mu \in \mathbb{R}$ and $\sigma > 0$. Denote by $\mathbb{Q}^*$ the unique equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$ on $\mathcal{F}_T$.

(a) Consider the *discounted* payoff

$$H := \max\left(S^1_T, \left(S^1_T\right)^3\right)$$

and denote by V^H_t its discounted arbitrage-free price at time $t \in [0, T]$. Prove that

$$V^H_t \geq \max\left(S^1_t, \left(S^1_t\right)^3\right).$$

(b) Consider the *undiscounted* payoff

$$\widetilde{Y} := \sqrt{\widetilde{S}^1_T \widetilde{S}^1_{T/2}}.$$

Compute its arbitrage-free price $V^{\widetilde{Y}}_0$ at time 0.

Possible hint: If $X \sim \mathcal{N}(0, 1)$, then $\mathbb{E}\left[e^{tX}\right] = e^{\frac{1}{2}t^2}$ for all $t \geq 0$.