ETH Zürich HS 2014
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 2

Please hand in by Wednesday, 01/10/2014, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 2-1. Consider a financial market $(\tilde{S}^0, \tilde{S}^1)$ with time horizon $T \geq 2$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $\tilde{S}^1_0 = 1$ and $\tilde{S}^1_k > 0$ $\mathbb{P}$-a.s. for all $k = 0, \ldots, T$. Fix thresholds $0 < \ell < 1 < u$ and define

$$\sigma := \inf\{k = 0, \ldots, T : S^1_k \leq \ell\} \wedge T, \tag{1}$$

$$\tau := \inf\{k = \sigma, \ldots, T : S^1_k \geq u\} \wedge T, \tag{2}$$

where we agree that $\inf \emptyset = +\infty$. Moreover, for $k = 0, \ldots, T$ define

$$\vartheta_k := \mathbf{1}_{\{\sigma < k \leq \tau\}}\,. \tag{3}$$

Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{0 \leq k \leq T}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_k = \sigma\Big(\{\tilde{S}^1_i,\ i \leq k\}\Big)$.

(a) Show that σ and τ are *stopping times*, i.e. that for all $k = 0, \ldots, T$, we have

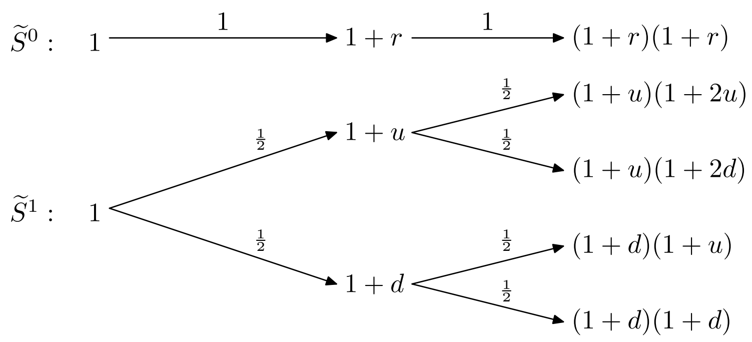
$$\{\sigma \leq k\},\ \{\tau \leq k\} \in \mathcal{F}_k\,. \tag{4}$$

(b) Show that ϑ is a real-valued predictable process with $\vartheta_0 = \vartheta_1 = 0$.

(c) Construct φ^0 such that $\varphi := (\varphi^0, \vartheta)$ is a self-financing strategy with $V_0(\varphi) = 0$ and derive a formula for the (discounted) value process $V(\varphi)$ only involving the discounted stock price S^1 and the stopping times σ and τ.

(d) Describe the trading strategy φ in words.

Exercise 2-2. Consider a financial market $(\tilde{S}^0, \tilde{S}^1)$ given by the following trees, where the numbers beside the branches denote transition probabilities:



Intuitively, this means that the volatility of $\tilde{S}^1$ increases after a stock price increase in the first period. Assume that $u, r \geq 0$ and $-0.5 < d \leq 0$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes $\tilde{S}^0$ and $\tilde{S}^1$ such that $\tilde{S}^1_k = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$.

(b) For which values of u and d are Y_1 and Y_2 *uncorrelated*?

(c) For which values of u and d are Y_1 and Y_2 *independent*?

(d) For which values of u, r and d is the discounted stock process S^1 a $\mathbb{P}$-martingale?

Exercise 2-3. Let $(\tilde{S}^0, \tilde{S}^1)$ be an *i.i.d. returns model* with $T = 3$ and assume that $\tilde{S}_0^1 = 1$. Moreover, assume that $r = 0.01$ and that $\log Y_k$ is *two-sided exponentially distributed* with parameter $\lambda \in (0, \infty)$, i.e. the probability distribution function (pdf) of $\log Y_1$ is given by $f(y) = \frac{\lambda}{2} \exp(-\lambda |y|)$, $y \in \mathbb{R}$. Define σ, τ, ϑ and φ as in Exercise 2-1 and let $\ell = 0.5$ and $u = 1.5$.

(a) For $\lambda = 2$, calculate $\mathbb{P}[\sigma = 1]$ and $\mathbb{P}[\tau = 2]$.

(b) Show that the trading strategy φ is *admissible* for all $\lambda \in (0, \infty)$ and that the trading strategy $-\varphi$ is *not* admissible for any $\lambda \in (0, \infty)$.

Hint. You may use that $V_k(\varphi) = S^1_{\tau \wedge k} - S^1_{\sigma \wedge k}$ for $k = 0, \ldots, 3$.

(c) For which $\lambda \in (0, \infty)$ is the discounted stock price S^1 a $\mathbb{P}$-martingale?

Exercise 2-4. Consider the market of Exercise 2-1. Assume, that the interest rate r is equal to 0, the riskless asset's value is therefore $\tilde{S}_k^0 \equiv 1$, $\forall\ 0 \leqslant k \leqslant T$, and that the price of the risky asset follows a binomial model :

$$\tilde{S}_k^1 = S_k^1 = \prod_{i=1}^{k} Y_i,$$

where the $(Y_i)_{1 \leqslant i \leqslant T}$ are i.i.d. random variables taking values in $\{1 + y_d, 1 + y_u\}$, the two values having probability $\frac{1}{2}$.

Let $T = 100$, $y_d = -0.1$, $y_u = 0.1$, $u = 1.2$, $l = 0.9$.

Use the software R to simulate the following :

(a) a path of the risky asset price,

(b) for this path, the corresponding self-financing strategy defined in Exercise 2-1 c),

(c) the value process of this strategy.

(d) Finally plot these different processes on the same graph.