ETH Zürich HS 2014
D-MATH
Prof. J. Teichmann

Coordinator
Thomas Cayé

Mathematical Foundations For Finance

Exercise Sheet 3

Please hand in by Wednesday, 8/10/2014, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 3-1. Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *binomial model*. More precisely, the price processes of the assets are defined as follows :

$$\widetilde{S}^0_k = (1+r)^k \qquad \text{for } k \geqslant 0$$
$$\frac{\widetilde{S}^1_{k+1}}{\widetilde{S}^1_k} = Y_{k+1} \qquad \text{for } k \geqslant 0,$$

where the Y_k's are i.i.d, taking value $1+u$ with probability $p \in (0,1)$ and $1+d$ with probability $1-p$. Assume furthermore that $u > d$.

(a) Suppose that $r \leq d$. Show that in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity.*

(b) Suppose that $r \geq u$. Show that also in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* again by explicitly constructing an *arbitrage opportunity.*

Exercise 3-2. Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *trinomial model*. This is like a binomial model a special case of a *multinomial model*, and the distribution of Y_k under $\mathbb{P}$ is given by

$$Y_k = \begin{cases} 1+d & \text{with probability } p_1 \\ 1+m & \text{with probability } p_2 \\ 1+u & \text{with probability } p_3 \end{cases}$$

where p_1, p_2, $p_3 > 0$, $p_1 + p_2 + p_3 = 1$ and $-1 < d < m < u$. Here d, m and u are mnemonics for *down*, *middle* and *up*. Assume that $d = -0.01$, $m = 0.01$, $u = 0.03$ and $r = 0.01$.

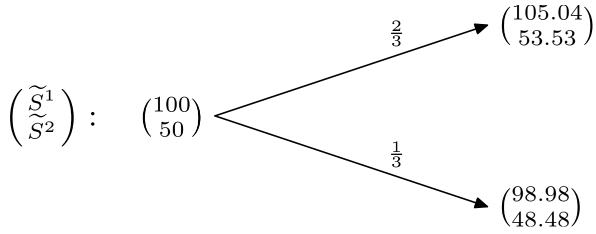
(a) For $T = 1$, give a parametrisation of all *equivalent martingale measures* (EMMs) for S^1.

Hint. A probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_1$ can be uniquely described by a *probability vector* $(q_1, q_2, q_3) \in (0,1)^3$, where $q_k = \mathbb{Q}[Y_1 = 1 + y_k]$, $k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.

(b) For $T = 2$, give a parametrisation of all *equivalent martingale measures* (EMMs) for S^1.

Hint. A probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_2$ can be uniquely described by four *probability vectors* (q_1, q_2, q_3), $(q_{j,1}, q_{j,2}, q_{j,3}) \in (0,1)^3$, $j = 1, 2, 3$, where $q_j = \mathbb{Q}[Y_1 = 1 + y_j]$ and $q_{j,k} = \mathbb{Q}[Y_2 = 1 + y_k \,|\, Y_1 = 1 + y_j]$, $j, k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.

Exercise 3-3. Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ consisting of a bank account and two stocks. The stock price movements of $\widetilde{S}^1$ and $\widetilde{S}^2$ are described by the following tree, where the numbers beside the branches denote transition probabilities.



Assume that the risk-free rate is given by $r = 0.01$.

(a) Show that the financial sub-markets $(\widetilde{S}^0, \widetilde{S}^1)$ and $(\widetilde{S}^0, \widetilde{S}^2)$ are free of arbitrage by constructing the equivalent martingale measures $\mathbb{Q}^1$ for S^1 and $\mathbb{Q}^2$ for S^2.

Hint. Write down the tree for the discounted stock prices S^1 and S^2.

(b) Show that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is not free of arbitrage by explicitly constructing an *arbitrage opportunity.*

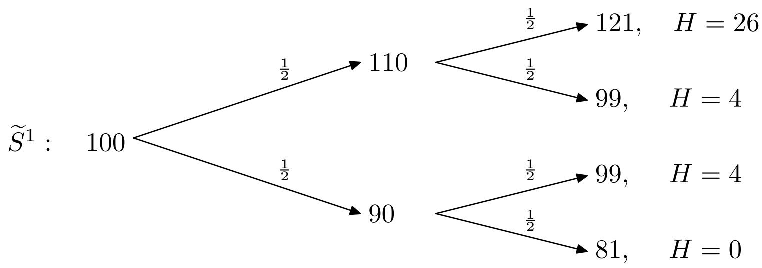
Hint. Calculate the expectation of S_1^2 under the equivalent martingale measure $\mathbb{Q}^1$ for S^1.

(c) By which number do you have to replace 105.04 in the stock price movement of $\widetilde{S}^1$ so that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is free of arbitrage?

Exercise 3-4. We consider a two periods binomial model, with two assets: one riskless asset with zero interest rate whose price is $S^0 \equiv \widetilde{S}^0 \equiv 1$, and a risky asset with price process given below. In this market we consider as well a call option on the risky asset with payoff H given by:

$$H = \left(S_2^1 - K\right)^+.$$

We take here $K = 95$. The value of the option at maturity is given on the tree.



We want to find a hedging portfolio for this option, i.e. we want to trade dynamically in the stock to replicate perfectly the contingent claim. In other word, we want to find a self-financing strategy $\phi \hat{=} (V_0, \theta)$ with initial value V_0 such that $V_2(\phi) = H$. We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\{0,1,2\}}$ generated by the risky asset (we assume $\mathcal{F}_0$ trivial).

(a) Find the self-financing replicating portfolio.

(b) Set the linear system that characterize the replicating strategy and use R to solve it.

(c) Simulate the price process of the risky asset and the self-financing portfolio that replicates the option.