ETH Zürich HS 2014
D-MATH
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# Mathematical Foundations For Finance

## Exercise Sheet 4

*Please hand in by Wednesday, 15/10/2014, 13:00, into the assistant's box next to office HG E 65.2.*

**Exercise 4-1.** Let $(\tilde{S}^0, \tilde{S}^1)$ be a *binomial model* and assume that $T = 1$, $u > r > 0$ and $-1 < d < 0$. For $K > 0$ define the functions $C(\bullet, K)$ and $P(\bullet, K) : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$C(x,K) := (x-K)^+ := \max(0, x-K) \quad \text{and} \quad P(x,K) := (K-x)^+ := \max(0, K-x)\,.$$

In financial terms, $C(\bullet, K)$ is the payoff function of a (long) *European call option with strike* $K$ and $P(\bullet, K)$ is the payoff function of a (long) *European put option with strike* $K$.

(a) Construct a self-financing strategy $\varphi^{C(K)} \mathrel{\hat{=}} (V_0^{C(K)}, \vartheta^{C(K)})$ such that

$$V_1(\varphi^{C(K)}) = \frac{C(\tilde{S}_1^1, K)}{1+r} \quad \mathbb{P}\text{-a.s.}\,.$$

(b) Construct a self-financing strategy $\varphi^{P(K)} \mathrel{\hat{=}} (V_0^{P(K)}, \vartheta^{P(K)})$ such that

$$V_1(\varphi^{P(K)}) = \frac{P(\tilde{S}_1^1, K)}{1+r} \quad \mathbb{P}\text{-a.s.}\,.$$

*Hint.* Both (a) and (b) reduce to solving two linear equations.

(c) Prove the *put-call parity*:

$$V_0^{P(K)} + S_0^1 = V_0^{C(K)} + \frac{K}{1+r}. \tag{$*$}$$

Give an economic interpretation of $(*)$.

(d) Compute the limits $\lim_{K\to\infty} V_0^{C(K)}$, $\lim_{K\to 0} V_0^{C(K)}$, $\lim_{K\to\infty} V_0^{P(K)}$ and $\lim_{K\to 0} V_0^{P(K)}$.

**Exercise 4-2.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and as usual $L^0_+$ the family of all nonnegative random variables. We say that $S$ satisfies $(NA')$ if for every self-financing strategy $\varphi$ with initial capital zero, i.e., $V_0(\varphi) = 0$, we have $V_T(\varphi) \notin L^0_+ \backslash \{\mathbf{0}\}$. Prove that $(NA)$ implies $(NA')$ by following the steps below.

(a) Start with a self-financing strategy $\varphi$ with $V_0(\varphi) = 0$ $\mathbb{P}$-a.s. and $V_T(\varphi) \in L^0_+ \setminus \{\mathbf{0}\}$. Identify $\varphi$ as in the course with $(0, \vartheta)$ and show that we are done if $G(\vartheta)$ is nonnegative.

(b) Now suppose that $\mathbb{P}[G_n(\vartheta) < 0] > 0$ for some $n \in \{1, \dots, T-1\}$. Define

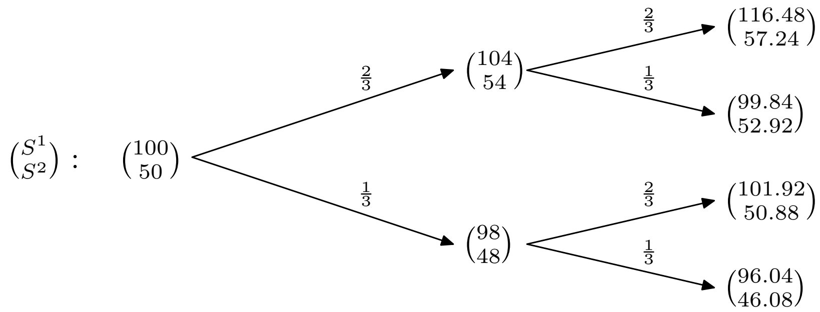
$$\begin{aligned} n_0 &:= \max\{n : \mathbb{P}[G_n(\vartheta) < 0] > 0\}, \\ A &:= \{G_{n_0}(\vartheta) < 0\}, \\ \vartheta'_n &:= \begin{cases} 0 & \text{if } n \le n_0 \\ \vartheta_n \mathbb{1}_A & \text{if } n > n_0 \end{cases}. \end{aligned}$$

Prove that $\vartheta'$ is predictable.

(c) Denote by $\varphi'$ the self-financing strategy corresponding to $(0, \vartheta')$. Prove that $\varphi'$ is 0-admissible.

(d) Show that $V_T(\varphi') \in L^0_+ \setminus \{\mathbf{0}\}$ and explain why this completes the proof.

**Exercise 4-3.** Consider a financial market $(\tilde{S}^0, \tilde{S}^1, \tilde{S}^2)$ consisting of a bank account and two stocks. The movements of the *discounted* stock prices $S^1$ and $S^2$ are described by the following tree, where the numbers beside the branches denote transition probabilities.




Assume that the risk-free rate is given by $r = 0.01$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, random variables $Y_k^1$ and $Y_k^2$, $k = 1, 2$ and adapted stochastic processes $\tilde{S}^0$, $\tilde{S}^1$ and $\tilde{S}^2$ such that $\tilde{S}_k^1 = \tilde{S}_0^1 \prod_{j=1}^k Y_j^1$ and $\tilde{S}_k^2 = \tilde{S}_0^2 \prod_{j=1}^k Y_j^2$ for $k = 0, 1, 2$.

(b) Show that the financial sub-markets $(\tilde{S}^0, \tilde{S}^1)$ and $(\tilde{S}^0, \tilde{S}^2)$ are free of arbitrage by constructing equivalent martingale measures $\mathbb{Q}^1$ for $S^1$ and $\mathbb{Q}^2$ for $S^2$.

(c) Show that the market $(\tilde{S}^0, \tilde{S}^1, \tilde{S}^2)$ is not free of arbitrage by explicitly constructing an *arbitrage opportunity.*

*Hint.* Calculate $\mathbb{E}_{\mathbb{Q}^1}\left[S_1^2 \,\middle|\, \mathcal{F}_0\right]$ and $\mathbb{E}_{\mathbb{Q}^1}\left[S_2^2 \,\middle|\, \mathcal{F}_1\right]$.

**Exercise 4-4.** We consider a classical binomial model with time horizon $T$ as defined in the lecture notes p 16. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider two assets, one riskless with price $S^0 \equiv 1$ and a risky asset whose price can go up by a factor $1 + u$ or down by a factor $1 + d$ during each period, both with a strictly positive probability. We assume $u = -d$, and $u > 0$. Hence, by corollary 2.1.4 (p 32), the market is free of arbitrage. We want to study and simulate the doubling strategy. It is the self-financing strategy $\phi \hat{=} (0, \theta)$ with $\theta$ defined as follows :

$$\theta_k = \frac{1}{S_{k-1}} 2^{k-1} \mathbb{1}_{\{k \leqslant \tau\}},$$

where $\tau = \inf\{k \mid Y_k = 1 + u\} \wedge T$. In other words, as long as the trader loses money on the market, he puts more money in stocks, so as to make an overall profit if he the stock goes up by $1 + u$.

(a) Simulate the price processes and the doubling strategy for $T = 200$, $u = 0.05$, $p = \dfrac{1}{2}$, $S_0^1 = 1$.

(b) Compute the value process of the portfolio.

(c) Use Monte-Carlo simulations to get the experimental distribution of the final wealth of the portfolio. What can you say about the strategy ? Is it a reasonable strategy to perform ? Can you see that from your simulations ?