ETH Zürich HS 2014
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 5

Please hand in by Wednesday, 22/10/2014, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 5-1. Consider the trinomial model with parameters

$$T = 2,\ r = 4\%,\ y_d = -12\%,\ y_m = 1\%,\ y_u = 10\%\,.$$

Find all equivalent martingale measures for $\frac{\tilde{S}^1}{\tilde{S}^0}$.

Hint. Describe the equivalent martingale measures in terms of transition probabilities.

Exercise 5-2. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_k)_{k=0,\dots,T})$ be a filtered probability space and $S = (S_k)_{k=0,\dots,T}$ a discounted price process. Show that the following are equivalent:

(a) S satisfies (NA).

(b) For each $k = 0, \dots, T-1$, the one-period market (S_k, S_{k+1}) on $(\Omega, \mathcal{F}_{k+1}, \mathbb{P}, (\mathcal{F}_k, \mathcal{F}_{k+1}))$ satisfies (NA).

Give an economic interpretation of this result.

Hint. Prove the contraposition of the direction "(b) $\Rightarrow$ (a)". Argue via induction on T.

Exercise 5-3. Let $(\widetilde{S}^0, \widetilde{S}^1)$ be an *arbitrage-free* financial market with time horizon T and assume that the bond satisfies $\widetilde{S}^0_k = (1+r)^k$ for $k = 0, \dots, T$ with $r \geq 0$. Denote the set of all equivalent martingale measures for S^1 by $\mathbb{P}_e(S^1)$. Fix $K > 0$ and let $k \in \{1, \dots, T\}$. The payoff of a *European call option* on $\widetilde{S}^1$ with strike K and maturity k is denoted by C^E_k and given by

$$C^E_k = \big(\widetilde{S}^1_k - K\big)^+\,,$$

whereas the payoff an *Asian call option* on $\widetilde{S}^1$ with strike K and maturity k is denoted by C^A_k and given by

$$C^A_k := \left(\frac{1}{k}\sum_{j=1}^{k} \widetilde{S}^1_j - K\right)^+\,.$$

(a) Fix $\mathbb{Q} \in \mathbb{P}_e(S^1)$. Show that the function $\{1, \dots, T\} \to \mathbb{R}^+$, $k \mapsto \mathbb{E}_{\mathbb{Q}}\left[\frac{C^E_k}{\widetilde{S}^0_k}\right]$ is increasing.

Hint. Use *Jensen's inequality* for conditional expectations.

(b) Fix $\mathbb{Q} \in \mathbb{P}_e(S^1)$. Show that for all $k = 1, \dots, T$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{C^A_k}{\widetilde{S}^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^{k} \mathbb{E}_{\mathbb{Q}}\left[\frac{C^E_j}{\widetilde{S}^0_j}\right]\,.$$

(c) Fix $\mathbb{Q} \in \mathbb{P}_e(S^1)$. Deduce that for all $k = 1, \dots, T$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{C^A_k}{\widetilde{S}^0_k}\right] \leq \mathbb{E}_{\mathbb{Q}}\left[\frac{C^E_k}{\widetilde{S}^0_k}\right]\,.$$

Please see next page!

Exercise 5-4. We want to check "experimentally" that the European call price is an increasing function of the claim maturity, as well as the last inequality of Exercise 5-3. Let us consider a binomial model, with $u = -d = 0.01$, $r = 0$, $\tilde{S}_0^1 = S_0^1 = 100$ and maturity $T = 100$. The unique risk-neutral probability in this model is the probability measure that assigns a probability of $\frac{1}{2}$ to both the up and the down moves. The returns (Y_i's) are here independent.

(a) Simulate the discounted price process and the payoff of these two options: the Asian option with maturity $T = 100$ and strike $K = 100$, and the European option with same maturity and strike.

(b) To check the formula of Exercise 5-3 a) and c) simulate these two options payoffs at maturity 100 000 times, and compute their experimental mean. For the European option plot the curve of the experimental mean payoff as a function of maturity.