Coordinator
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Mathematical Foundations For Finance

Exercise Sheet 6

Please hand in by Wednesday, 29/10/2014, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 6-1. Consider a discounted model for a financial market which is free of arbitrage. Suppose we enlarge the financial market S by introducing a new financial instrument which can be bought or sold at price $a \in \mathbb{R}$ at time $t = 0$ and yields a random cash flow f at time $t = T$. This means that we are allowed to trade dynamically in S, i.e. we can buy and sell stocks at prices S_t at all times $t = 0, \dots, T$, but we can only trade statically in f, i.e. we can only buy or sell the new instrument per unit at price a and hold this position until T. We denote this market by $(S, (f, a))$. The goal of this exercise is to introduce and study the possible arbitrage-free prices a for f.

Formally, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,\dots,T}$ and assume for simplicity that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $S = (S_t)_{t=0,\dots,T}$ be a discounted model for a financial market which is free of arbitrage, i.e. satisfies (NA), and let $f \in L^0_+(\mathcal{F}_T)$ be a contingent claim. For $a \in \mathbb{R}$, we first want to define what is meant by an *arbitrage opportunity for the enlarged market* $(S, (f, a))$. To do so, we consider all possible positions at time T that one can obtain via dynamic-static trading with respect to S and f with zero initial investment. A moment's reflection reveals that these positions are those of the form

$$G_T(\vartheta) + \lambda(f - a)\,, \tag{1}$$

where ϑ is a predictable process representing a self-financing trading strategy $\varphi = (0, \vartheta)$ and $\lambda \in \mathbb{R}$ represents the number of units of f bought or sold at $t = 0$. In accordance with the usual definition of (NA), we say that the market $(S, (f, a))$ is *free of arbitrage* if there exists no $\varphi = (0, \vartheta)$ and no $\lambda \in \mathbb{R}$ such that

$$G_T(\vartheta) + \lambda(f - a) \geq 0 \quad \mathbb{P}\text{-a.s.and} \quad \mathbb{P}\left[G_T(\vartheta) + \lambda(f - a) > 0\right] > 0\,. \tag{2}$$

We say that a is an *arbitrage-free price for* f if the extended market $(S, (f, a))$ is free of arbitrage. Similarly to Proposition 2.1.1 part 4) in the lecture notes, we define $\mathcal{G}^{f,a}$ to be the set of all random variables of the form (1). We then obtain the equivalent concise definition that a is an *arbitrage-free price* for the enlarged market $(S, (f, a))$ if and only if

$$\mathcal{G}^{f,a} \cap L^0_+(\mathcal{F}_T) = \{\mathbf{0}\}\,.$$

(a) Show that if a is an arbitrage-free price for f, then there exists an equivalent probability measure $\mathbb{Q} \approx \mathbb{P}$ such that

$$\mathbb{E}_{\mathbb{Q}}\left[g\right] = 0 \quad \text{for all } g \in \mathcal{G}^{f,a}\,.$$

Conclude that

(i) $\mathbb{Q}$ is an equivalent martingale measure for S, i.e. $\mathbb{Q} \in \mathbb{P}_e(S)$.

(ii) $\mathbb{E}_{\mathbb{Q}}\left[f\right] = a$.

Hint. Modify the proof of Theorem 2.2.1 in the lecture notes appropriately. You need not copy the whole proof; just indicate where and which changes need to be made.

In the next step, we want to characterise the arbitrage-free prices for f. To that end, we define the two quantities

$$\overline{\pi}(f) := \sup_{\mathbb{Q} \in \mathbb{P}_e(S)} \mathbb{E}_{\mathbb{Q}}\left[f\right], \quad \underline{\pi}(f) := \inf_{\mathbb{Q} \in \mathbb{P}_e(S)} \mathbb{E}_{\mathbb{Q}}\left[f\right]\,.$$

(b) Prove that if $\underline{\pi}(f) = \overline{\pi}(f)$, then f is attainable in the market S at price $a = \underline{\pi}(f) = \overline{\pi}(f)$.

Hint. Use Theorem 3.1.2 in the lecture notes.

(c) Prove that if $\underline{\pi}(f) < \overline{\pi}(f)$, then a is an arbitrage-free price if and only if

$$\underline{\pi}(f) < a < \overline{\pi}(f)\,.$$

Hint. Use Theorem 3.1.2 in the lecture notes and part (a).

Exercise 6-2. Consider the trinomial model with $r = 0.05$ and $T = 1$. Suppose that the evolution of $\left(\widetilde{S}^0, \widetilde{S}^1\right)$ is given by

$$\widetilde{S}^1_0 = S^1_0 = s_0 = 80, \quad \widetilde{S}^1_1 = \begin{cases} 120 & \text{with probability } 0.2 \\ 90 & \phantom{\text{with probability }} 0.3 \\ 60 & \phantom{\text{with probability }} 0.5 \end{cases}, \text{ and } \widetilde{S}^0_k = (1+r)^k\,, \text{ for } k \in \{0,1\}.$$

(a) Compute the set of all arbitrage-free prices for the European call option $\widetilde{H} = (\widetilde{S}^1_1 - 80)^+$.

(b) Find the set of all attainable contingent claims.

(c) Is it possible to replicate the previous call option by a self-financing portfolio?

Exercise 6-3. Consider the trinomial model with $r = 0.2$ and $T = 1$. Suppose that the evolution of $\widetilde{S}^1$ is given by

$$S^1_0 = s_0, \qquad \widetilde{S}^1_1 = \begin{cases} s_0(1+u) \\ s_0(1+m) \\ s_0(1+d) \end{cases},$$

with $u = 0.6, m = r, d = -0.2$. Let $\widetilde{H} = \mathbb{1}_{\{S^1_1 < s_0\}}$ and denote by $\mathcal{P} = \mathbb{P}_e(S)$ the set of all equivalent martingale measures for the discounted price process S^1.

(a) Compute $V_0 := \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[H]$ for the discounted payoff H.

(b) Show that $V_0 = \mathbb{E}_{\tilde{\mathbb{Q}}}[H]$ for some martingale measure $\tilde{\mathbb{Q}}$, but $\tilde{\mathbb{Q}}$ is not equivalent to $\mathbb{P}$.

(c) Deduce that H is not attainable.

Exercise 6-4. We consider a binomial market model with N periods on a period of time of length T. The riskless asset grows at a rate $r = \frac{RT}{N}$, where R is the (constant) instantaneous interest rate, and the risky asset's price goes up by a factor $1 + u$ and down by a factor $1 + d$ such that

$$\log\left(\frac{1+u}{1+r}\right) = -\log\left(\frac{1+d}{1+r}\right) = \sigma\sqrt{\frac{T}{N}},$$

for some constant σ. The starting values (at time t=0) of both the assets is 1 $\mathbb{P}$-a.s. The unique equivalent martingale measure $\mathbb{P}^*$ for S^1 is such that the $(Y_i)_{i\in\{1,2\ldots,N\}}$ are i.i.d. and given by

$$\mathbb{P}^*\left[Y_i = 1+d\right] = 1 - \mathbb{P}^*\left[Y_i = 1+u\right] = \frac{u-r}{u-d} = p^*$$

We study the limiting case for $N \to \infty$.

Please see next sheet!

(a) Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables of the form :

$$Z_n = \sum_{i=1}^{n} X_i^n$$

for $n \in \mathbb{N}$, $X_i^n \in \{-\sigma\sqrt{\frac{T}{n}},\ \sigma\sqrt{\frac{T}{n}}\}$ and the variables $(X_i^n)_{i\in\{1,2,\ldots n\}}$ are independent identically distributed with mean μ_n. The constants μ_n are such that $\lim_{n\to\infty} n\mu_n = \mu$.

Prove that the sequence $(Z_n)_{n\in\mathbb{N}}$ converges in law to a gaussian random variable with mean μ and variance $\sigma^2 T$.

Hint. Use the fact that pointwise convergence of the characteristic functions of a sequence of random variables (if the limiting function ϕ is continuous at 0) implies the convergence in law of this sequence of random variables to a random variable whose characteristic function is ϕ.

(b) We consider a European put option, with strike K and maturity T. Show that its value at time 0 is given by

$$V_0^{P,N} = \mathbb{E}^*\left[\left(\frac{K}{(1+r)^N} - S_0^1 \exp\left(Z_N\right)\right)^+\right],$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$, and Z_N is a random variable that you will define.

(c) Use part a) to prove the following asymptotic price :

$$\lim_{N\to\infty} V_0^{P,N} = Ke^{-RT}\Phi(-d_2) - S_0^1\Phi(-d_1),$$

where $d_1 = \dfrac{\log\left(\frac{S_0}{K}\right) + RT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$ and Φ is the cumulative distribution function of a standard normal random variable.

Hint. Use the value of p^* and of u to prove that $\lim_{N\to\infty} N\mathbb{E}^*\left[\log\left(\frac{Y_i}{1+r}\right)\right] = -\frac{\sigma^2 T}{2}$

Remark. With the put-call parity formula, one gets easily the asymptotic price of a call for large N. At the end of the course you will see as well that this limit is the price that one obtains in the Black-Scholes model.

(d) Take $T = 100$, $N = 1000$, $S_0^1 = 100$, $K = 80$, $R = 0.01$. The put option with strike K and maturity T is sold on the market for 10. Back out with R the parameter σ and then u.

(e) For this last question, let us use real life figures. We consider the Exchange Traded Fund S&P 500. Look for the current price of this fund, and the price of the put with strike 195$ and maturity 19th december 2015 (the identifier of the option is SPY151219P00195000). Use the approximation $R = 0$ (not very far from the actual over-night LIBOR rates). Back out the parameter σ and u from the formula obtained in c).