ETH Zürich HS 2014
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 7

Please hand in by Wednesday, 05/11/2014, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 7-1. Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with $\widetilde{S}^1_0 := 1$ and $u > r > d > -1$. Denote by $(\widehat{S}^0, \widehat{S}^1)$ the market discounted with $\widetilde{S}^1$, i.e.

$$\widehat{S}^0 := \frac{\widetilde{S}^0}{\widetilde{S}^1} \quad \text{and} \quad \widehat{S}^1 := \frac{\widetilde{S}^1}{\widetilde{S}^1} \equiv 1\,.$$

(a) Show that there exists a unique equivalent martingale measure $\mathbb{Q}^{**}$ for $\widehat{S}^0$.

(b) Let $\mathbb{Q}^*$ be the unique equivalent martingale measure for S^1. Show that the density of $\mathbb{Q}^{**}$ with respect to $\mathbb{Q}^*$ on $\mathcal{F}_T$ is given by

$$\frac{\mathrm{d}\mathbb{Q}^{**}}{\mathrm{d}\mathbb{Q}^*} = S^1_T\,.$$

Hint. Use Corollary 2.1.4 in the lecture notes.

(c) Show that for an *undiscounted* payoff $\widetilde{H} \in L^0_+(\mathcal{F}_T)$ we have

$$\widetilde{S}^0_k\, \mathbb{E}_{\mathbb{Q}^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\,\middle|\,\mathcal{F}_k\right] = \widetilde{S}^1_k\, \mathbb{E}_{\mathbb{Q}^{**}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\,\mathcal{F}_k\right], \quad k = 0, \dots, T\,.$$

This formula shows that the risk-neutral pricing method is invariant under a so-called *change of numéraire.*

Hint. Use *Bayes' formula* (Lemma 2.3.1 2) in the lecture notes).

Exercise 7-2. An *American option* with maturity T and payoff process $U = (U_k)_{k=0,\dots,T}$, where U is an adapted process, is a contract between buyer and seller where the buyer has the right to stop the contract at any time $0 \le k \le T$ and then to receive the (discounted) payoff U_k. The buyer is allowed to choose as exercise time for the option any stopping time with values in $\{0, \dots, T\}$. The goal of this exercise is to analyze the corresponding *arbitrage-free price* of an American option. With some effort, one can show that the *arbitrage-free price process* $\overline{V} = (\overline{V}_k)_{k=0,\dots,T}$ for an American option can be expressed by the backward recursive scheme

$$\overline{V}_T = U_T,$$
$$\overline{V}_k = \max\left\{U_k, \mathbb{E}_{\mathbb{Q}}\left[\overline{V}_{k+1}\,\middle|\,\mathcal{F}_k\right]\right\} \quad \text{for } k = 0, \dots, T-1\,, \tag{1}$$

where $\mathbb{Q}$ is an equivalent martingale measure for the considered market.

(a) Give an economic argument why (1) is a reasonable.

(b) Show that $\overline{V}$ is the smallest $\mathbb{Q}$-supermartingale dominating U, i.e., show that

(i) $\overline{V}$ is a $\mathbb{Q}$-supermartingale such that $\overline{V}_k \ge U_k$ $\mathbb{P}$-a.s. for all $k = 0, \dots, T$.

(ii) if V' is a $\mathbb{Q}$-supermartingale such that $V'_k \ge U_k$ $\mathbb{P}$-a.s. for all $k = 0, \dots, T$, then $V'_k \ge \overline{V}_k$ $\mathbb{P}$-a.s. for all $k = 0, \dots, T$.

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(c) Assume now that $r > 0$ so that the bank account is strictly increasing.

(i) Show that in the *put option* case, i.e., $U_j = \frac{1}{(1+r)^j}(\widetilde{K} - \widetilde{S}^1_j)^+$, the price of an American option at time 0 is greater that the price of a European option, for large enough strikes $\widetilde{K}$, i.e.,

$$\overline{V}_0 > V_0^{\widetilde{P}^{\widetilde{K}}_T},$$

for $\widetilde{K}$ large enough, where $V_0^{\widetilde{P}^{\widetilde{K}}_T}$ denotes the discounted price at time 0 of a European put option with maturity T and strike price $\widetilde{K}$.

(ii) Show that in the *call option* case, i.e., $U_j = \frac{1}{(1+r)^j}(\widetilde{S}^1_j - \widetilde{K})^+$, the price of the American call option and the European call option coincide. This means, show that

$$\overline{V}_0 = V_0^{\widetilde{C}^{\widetilde{K}}_T},$$

where $V_0^{\widetilde{C}^{\widetilde{K}}_T}$ denotes the price at time 0 of an European call option with maturity T and strike price $\widetilde{K}$.

Exercise 7-3. We consider an American option with maturity T and payoff process $Z = (Z_k)_{k=0,...,T}$ on a complete market with pricing measure $\mathbb{Q}$. Assume Z is adapted (or consider the filtration generated by the payoff process). We want to prove that the price process of the American option is indeed given by the process $U = (U_k)_{k=0,...,T}$ defined as follows :

$$U_T = Z_T$$
$$U_k = \max\left(Z_k, \mathbb{E}\left[U_{k+1}|\mathcal{F}_n\right]\right), \text{ for } k \in \{0, 1, ..., T-1\}.$$

This process is called the Snell enveloppe of Z. It is the smallest supermartingale that dominates the process Z as proved in Exercise 7-2.

(a) Define the random variable $\sigma_0 = \inf\{n \geqslant 0 \mid U_n = Z_n\}$. Prove that it is a stopping time for the filtration generated by the payoff process Z.

(b) Prove that the stopped process $(U_{k\wedge\sigma_0})_{k\in\{0,1,...,T\}}$ is a martingale.

(c) Define for $k \in \{0, 1, ..., T\}$ the set $\mathcal{T}_{k,T}$ of all stopping times taking values in $\{k, k+1, ..., T\}$. Prove that σ_0 satisfies :

$$U_0 = \mathbb{E}\left[Z_{\sigma_0} \mid \mathcal{F}_0\right] = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}\left[Z_{\sigma_0} \mid \mathcal{F}_0\right]$$

and more generally :

$$U_k = \mathbb{E}\left[Z_{\sigma_k} \mid \mathcal{F}_k\right] = \sup_{\tau\in\mathcal{T}_{k,T}} \mathbb{E}\left[Z_{\sigma_k} \mid \mathcal{F}_k\right] \text{for } k \in \{0, 1, ..., T\},$$

where : $\sigma_k = \inf\{n \geqslant k \mid U_n = Z_n\}$.

Exercise 7-4. In this exercise we want to compare the price of a European call option and an American call option over time (and verify that the price process of these two options are indeed the same). We consider a binomial model with $T = 4$ periods, $S^1_0 = 100$, $K = 80$, $u = -d = 0.1$ and $r = 0$.

(a) Simulate the binomial market price tree and compute the option prices at each node as well as the replicating strategy.

Please see next sheet!

(b) Modify your code to compute the price of a European and an American put option over time as well as their replicating strategies. What do you observe ?