ETH Zürich HS 2014
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 8

Please hand in by Wednesday, 12/11/2014, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 8-1. We work on a binomial model; the risky and riskless asset have the following price processes :

$$\widetilde{S_k^1} = \widetilde{S_0^1}\prod_{i=1}^k Y_i, \text{ and, } \widetilde{S_k^0} = (1+r)^k \text{ for } k \in \{0,1,...,T\},$$

where the Y_i's are i.i.d and take values in $\{1+d, 1+u\}$. Consider a contingent claim of the form $\widetilde{H} = \widetilde{h}(\widetilde{S}_T^1)$, where $\widetilde{h} : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function. For simplicity, we denote by $\mathbb{Q}$ the unique martingale measure for the discounted model S^1. By Corollary 2.2.3 in the lecture notes, this model is arbitrage-free and complete. Therefore, there exists a self-financing trading strategy $\varphi = (V_0^H, \vartheta)$ such that

$$\widetilde{V}_T^{\widetilde{H}} = \widetilde{S}_T^0\left(V_0^H + G_T(\vartheta)\right) = \widetilde{H} \quad \mathbb{P}\text{-a.s.}.$$

(a) Show that there exists a measurable function $\widetilde{v} : \{0,\dots,T\} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\widetilde{V}_k^{\widetilde{H}} = \widetilde{v}(k, \widetilde{S}_k^1) \quad \mathbb{P}\text{-a.s. for } k = 0,\dots,T.$$

Moreover, show that this value function $\widetilde{v}$ fulfills the recursive scheme

$$\begin{cases} \widetilde{v}(T,x) & = \widetilde{h}(x), \\ \widetilde{v}(k-1,x) & = \frac{q\widetilde{v}(k,x(1+u))+(1-q)\widetilde{v}(k,x(1+d))}{1+r} \quad \text{for } k = 1,\dots,T,\ x \in \mathbb{R}_+ . \end{cases}$$

(b) Show that there exists a measurable function $\widetilde{\xi} : \{1,\dots,T\} \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\vartheta_k = \widetilde{\xi}(k, \widetilde{S}_{k-1}^1) \quad \mathbb{P}\text{-a.s. for } k = 1,\dots,T.$$

Moreover, show that the function $\widetilde{\xi}$ is given by

$$\widetilde{\xi}(k,x) = \frac{\widetilde{v}(k,x(1+u)) - \widetilde{v}(k,x(1+d))}{(u-d)x}.$$

(c) If the function $\widetilde{h}$ is *convex*, show that for each k the function $x \mapsto \widetilde{v}(k,x)$ is convex as well. If $\widetilde{h}$ is increasing, show that $\widetilde{\xi} \geq 0$. What is the financial interpretation of the latter property?

Exercise 8-2. Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a one-period *trinomial model* on the canonical space and assume that $u > m > d > -1$, $r = m$ and $\widetilde{S}_0^1 = s_0 > 0$.

(a) Compute the set of all arbitrage-free prices for a *binary cash-or-nothing call option* with strike $s_0(1+r)$ whose payoff is given by

$$\widetilde{H}^b := \mathbb{1}_{\{\widetilde{S}_1^1 > s_0(1+r)\}}.$$

Show that $\widetilde{H}^b$ is not attainable in this market.

Hint: Use Theorem 3.1.2 in the lecture notes.

Please see next page!

(b) Let $\widetilde{H}^{C(K)}$ be a *European call option* with strike $K \geq 0$ whose payoff is given by

$$\widetilde{H}^{C(K)} := (\widetilde{S}_1^1 - K)^+.$$

Determine all $K \geq 0$ for which $\widetilde{H}^{C(K)}$ is attainable.

(c) Let $\widetilde{H} \in L^0_+(\mathcal{F}_1)$ be an *undiscounted* payoff that is attainable with $V_0^{\widetilde{H}} \neq 0$. Define the *return* of $\widetilde{H}$ by

$$R^{\widetilde{H}} := \frac{\widetilde{H} - V_0^{\widetilde{H}}}{V_0^{\widetilde{H}}} = \frac{\widetilde{H}}{V_0^{\widetilde{H}}} - 1.$$

Show that every equivalent martingale measure $\mathbb{Q}$ for S^1 satisfies

$$\mathbb{E}_{\mathbb{Q}}\left[R^{\widetilde{H}}\right] = r \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}\left[R^{\widetilde{H}}\right] = r - \mathrm{Cov}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}, R^{\widetilde{H}}\right],$$

where $\mathrm{Cov}_{\mathbb{P}}$ denotes the covariance under $\mathbb{P}$.

Exercise 8-3. Let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted multinomial model with $m \geq 3$ states. Assume that $T = 1$ and $y_1 < r < y_m$. Denote by $\mathbb{P}_e(S^1)$ the set of all equivalent martingale measures for S^1 on $\mathcal{F}_1$ and let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be a *convex* discounted payoff function.

(a) Show that for all $\mathbb{Q} \in \mathbb{P}_e(S^1)$, we have

$$C(S_0^1) \leq \mathbb{E}_{\mathbb{Q}}\left[C(S_1^1)\right] \leq \frac{y_m - r}{y_m - y_1} C\left(\frac{1+y_1}{1+r} S_0^1\right) + \frac{r - y_1}{y_m - y_1} C\left(\frac{1+y_m}{1+r} S_0^1\right).$$

In particular, show that either both inequalities are strict or both equalities. Give an economic interpretation of the above formula.

Hint. Distinguish the two cases that C is linear or not linear on $\left[\frac{1+y_1}{1+r} S_0^1, \frac{1+y_m}{1+r} S_0^1\right]$. Drawing a picture might be useful.

(b) Show that the upper bound in (a) is sharp, i.e.,

$$\sup_{\mathbb{Q} \in \mathbb{P}_e(S^1)} \mathbb{E}_{\mathbb{Q}}\left[C(S_1^1)\right] = \frac{y_m - r}{y_m - y_1} C\left(\frac{1+y_1}{1+r} S_0^1\right) + \frac{r - y_1}{y_m - y_1} C\left(\frac{1+y_m}{1+r} S_0^1\right).$$

(c) Suppose there exists $k \in \{2, \ldots, m-1\}$ with $y_k = r$. Show that the lower bound in (a) is sharp as well, i.e.,

$$\inf_{\mathbb{Q} \in \mathbb{P}_e(S^1)} \mathbb{E}_{\mathbb{Q}}\left[C(S_1^1)\right] = C(S_0^1).$$

(d) Is the payoff $H = C(S_1^1)$ attainable?

Exercise 8-4. In this exercise we want to compare the price of a European put option and an American put option over time (and verify that the price process of these two options can be different). We consider a binomial model with $T = 4$ periods, $S_0^1 = 100$, $K = 150$, $u = -d = 0.2$ and $r = 0.1$.

(a) Simulate the binomial market price tree and compute the option prices at each node as well as the replicating strategy.

(b) Change the strike to $K = 80$. What do you observe ?

(c) Change the strike to $K = 50$. What do you observe ?