ETH Zürich HS 2014
D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 9

Please hand in by Wednesday, 19/11/2013, 13:00, into the assistant's box next to office HG E 65.2.

Exercise 9-1. We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let W be a standard Brownian motion defined on this space, and $\mathbb{F} = (\mathcal{F}_t)_{\{t \geqslant 0\}}$ be the augmented filtration generated by the Brownian motion. Use Itô formula to write the following processes as stochastic integrals.

(a) $X_t^{(1)} = W_t^2$.

(b) $X_t^{(2)} = t^2 W_t^3$.

(c) $X_t^{(3)} = \exp(\mu t + \sigma W_t)$.

(d) $X_t^{(4)} = \cos(t + W_t)$.

(e) $X_t^{(5)} = \log(2 + \cos(W_t - t))$.

(f) Let X and Y be two continuous real-valued semimartingales. Define the process $Z = XY$. Apply Itô's formula to Z and write it as a sum of stochastic integrals.

Exercise 9-2. Let $W = (W_t)_{t \geq 0}$ be a Brownian motion (BM) defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

(a) $W^{(1)} := -W$ is a BM.

(b) $W_t^{(2)} := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^{(3)} := \alpha B + \sqrt{1 - \alpha^2} B'$ is a BM, where B and B' are two independent BMs and $\alpha \in (0, 1)$.

(d) Show that the independence of B and B' in (c) cannot be omitted, i.e., if B and B' are not independent then $W^{(3)}$ need not be a BM. Give two examples.

Exercise 9-3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with an augmented filtration $\mathbb{F} = (\mathcal{F})_{t \geqslant 0}$ generated by a Brownian motion W.

We consider a Bachelier model for the discounted stock, i.e. the discounted price process is given by:

$$S_t^1 = x + \sigma W_t.$$

Under $\mathbb{P}$, the discounted price process is a martingale.

Let $K \in \mathbb{R}$, compute the hedging strategy for the contingent claims $h(S_T^1)$ for the following payoff functions:

(a) $h(y) = y^3 - 3y$

(b) $h(y) = y^3$

(c) $h(y) = (K - y)^+$

(d) $h(y) = \mathbb{1}_{\{y \geqslant K\}}$

Please see next page!

Assume that $\mathbb{E}\left[h\left(S_T^1\right)\right] < \infty$ and that the function defined as:

$$u\left(y, s\right) = \mathbb{E}\left[h\left(y + \sigma W_s\right)\right]$$

is continuously differentiable in s, twice continuously differentiable in y and such that $\mathbb{E}\left[\int_0^T \frac{\partial u}{\partial y}\left(S_t^1, T - t\right)^2 dt\right] < \infty$.

Hint. Compute the function u, express the value process in terms of u, and then use Itô's formula to obtain a self-financing strategy.