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D-MATH
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Mathematical Foundations For Finance

Exercise Sheet 14 (with Solutions)

Exercise 14-1. Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual assumptions. Let $W = (W_t)_{t\in[0,T]}$ be a $(\mathbb{P}, \mathbb{F})$-Brownian motion and $N = (N_t)_{t\in[0,T]}$ an *independent* $(\mathbb{P}, \mathbb{F})$-Poisson process with parameter $\lambda > 0$. Consider a discounted stock price $S = (S_t)_{t\in[0,T]}$ defined by

$$S_t := \exp\left(\sigma W_t + \log(1+\kappa)N_t + \left(\mu - \frac{1}{2}\sigma^2 - \kappa\lambda\right)t\right),$$

where $\mu \in \mathbb{R}$, $\kappa > -1$, and $\sigma > 0$.

(a) Use Itô's formula to show that

$$\mathrm{d}S_t = S_{t-}\left(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \kappa\,\mathrm{d}\widetilde{N}_t\right), \quad S_0 = 1,$$

where $\widetilde{N}_t := N_t - \lambda t$ is the compensated Poisson process.

Hint. Define $S_t^c := e^{\sigma W_t + (\mu - 1/2\sigma^2)t}$ and $S_t^d := e^{\log(1+\kappa)N_t - \kappa\lambda t}$ so that $S = S^c S^d$.

(b) Define the strictly positive $(\mathbb{P}, \mathbb{F})$-martingale $Z := \mathcal{E}(-\mu/\sigma W)$ and the equivalent probability measure $\mathbb{Q}$ via $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} := Z_T$.

Argue in detail that N is a $(\mathbb{Q}, \mathbb{F})$-Poisson process with same parameter λ. Conclude that S is a local $(\mathbb{Q}, \mathbb{F})$-martingale with dynamics

$$\mathrm{d}S_t = S_{t-}\left(\sigma\,\mathrm{d}W_t^{\mathbb{Q}} + \kappa\,\mathrm{d}\widetilde{N}_t\right),$$

where $(W_t^{\mathbb{Q}})_{t\in[0,T]}$, $W_t^{\mathbb{Q}} := W_t + \mu/\sigma\ t$, is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion.

Hint. Recall the definition of a Poisson process relative to $(\mathbb{P}, \mathbb{F})$ and check the conditions separately. You may use the following fact:

A random variable X and a σ-field $\mathcal{G}$ are independent if and only if $\mathbb{E}[f(X)\,|\,\mathcal{G}] = \mathbb{E}[f(X)]$ for all bounded Borel functions $f : \mathbb{R} \to \mathbb{R}$.

(c) Let $\alpha \in \mathbb{R}$. Compute $\mathbb{E}_{\mathbb{Q}}[(S_T)^\alpha]$.

Solution 14-1. (a) Define (as given in the hint) the two auxiliary processes

$$S_t^c := \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right) \quad \text{and} \quad S_t^d := \exp\left(\log(1+\kappa)N_t - \kappa\lambda t\right). \tag{1}$$

The process S^c is a geometric Brownian motion. By writing $W_t^{(\sigma,\mu)} := \sigma W_t + \mu t$, we have that $S^c = \mathcal{E}(W^{(\sigma,\mu)})$ and hence by Itô's formula:

$$\mathrm{d}S^c = S^c\,\mathrm{d}W^{(\sigma,\mu)} = S^c(\sigma\,\mathrm{d}W + \mu\,\mathrm{d}t). \tag{2}$$

We notice that if $f : \mathbb{R} \to \mathbb{R}$ is in C^2, $\alpha, \beta \in \mathbb{R}$ and the semimartingale $X = (X_t)_{t\geq 0}$ is given by $X_t = \alpha t + \beta N_t$, then formula (6.1.7) in the lecture notes simplifies to

$$f(X_t) = f(0) + \alpha \int_0^t f'(X_{s-})\,\mathrm{d}s + \sum_{0<s\leq t} \Big(f(X_s) - f(X_{s-})\Big).$$

Indeed, using the formula at the bottom of page 89 in the lecture notes and the fact that the quadratic variation process of the compensated Poisson process $\widetilde{N}$ is N, we get: $[X]_t = \sum_{0<s\leq t} \beta^2 (\Delta N_s)^2 = \sum_{0<s\leq t} (\Delta X_t)^2$.
We obtain:

$$\begin{aligned} S_t^d &= 1 - \kappa\lambda \int_0^t S_{u-}^d\,\mathrm{d}u + \sum_{0<u\leq t} \Big(S_u^d - S_{u-}^d\Big) \\ &= 1 - \kappa\lambda \int_0^t S_{u-}^d\,\mathrm{d}u + \kappa \sum_{0<u\leq t} S_{u-}^d \Delta N_u \\ &= 1 - \kappa\lambda \int_0^t S_{u-}^d\,\mathrm{d}u + \kappa \int_0^t S_{u-}^d\,\mathrm{d}N_u \\ &= 1 + \kappa \int_0^t S_{u-}^d\,\mathrm{d}\widetilde{N}_u. \end{aligned}$$

One can also write in differential notation

$$\mathrm{d}S^d = \kappa S_-^d\,\mathrm{d}\widetilde{N}. \tag{3}$$

The next step is to apply the product rule to $S = S^c S^d$. To that end, we need to have a formula for $[S^c, S^d]$. By the formula for quadratic variations (see LN p. 86 bottom), we have

$$[S^c, S^d]_t = \int_0^t S_u^c S_{u-}^d\,\mathrm{d}[W^{(\sigma,\mu)}, \widetilde{N}]_u.$$

By definition of $[\bullet, \bullet]$ for semimartingales (see LN p. 89 bottom), we have

$$[W^{(\sigma,\mu)}, \widetilde{N}]_t = \sum_{0<u\leq t} \Delta W_u^{(\sigma,\mu)} \Delta N_u = 0$$

and consequently $[S^c, S^d] = 0$. Applying now the product rule to S finally yields

$$\begin{aligned} \mathrm{d}S &= S_-^d\,\mathrm{d}S^c + S^c\,\mathrm{d}S^d \\ &= S^c S_-^d(\sigma\,\mathrm{d}W + \mu\,\mathrm{d}t) + \kappa S^c S_-^d\,\mathrm{d}\widetilde{N} \\ &= S_-\left(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W + \kappa\,\mathrm{d}\widetilde{N}\right). \end{aligned}$$

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(b) We recall from Exercise 10-1 the definition of a Poisson process (here for finite time horizon).

A $(\mathbb{P}, \mathbb{F})$-*Poisson process with parameter* $\lambda > 0$ is a (real-valued) stochastic process $N = (N_t)_{t \in [0,T]}$ which is $\mathbb{F}$-adapted, starts at 0 (i.e. $N_0 = 0$ $\mathbb{P}$-a.s.) and satisfies the following two properties:

(PP1) For $0 \le t < t+h \le T$, the *increment* $N_{t+h} - N_t$ is independent (under $\mathbb{P}$) of $\mathcal{F}_t$ and is (under $\mathbb{P}$) *Poisson-distributed with parameter* λh, i.e.

$$\mathbb{P}[N_{t+h} - N_t = k] = \frac{(\lambda h)^k}{k!} e^{-\lambda h}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for $\mathbb{P}$-almost all ω, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant and $\mathbb{N}_0$-valued, and increases by jumps of size 1.

We check the conditions separately. Since N is a $(\mathbb{P}, \mathbb{F})$-Poisson process and since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, N fulfills (PP2) under the measure $\mathbb{Q}$, $N_0 = 0$ $\mathbb{Q}$-a.s. (trivially) and N is $\mathbb{F}$-adapted. It remains to check (PP1).

To that end, we fix $0 \le t < t+h \le T$ and and bounded Borel function $f : \mathbb{R} \to \mathbb{R}$. Using the ($\mathbb{P}$-)independence of N and W, that $W_{t+h} - W_t$ (or rather Z_{t+h}/Z_t) and $N_{t+h} - N_t$ are ($\mathbb{P}$)-independent of $\mathcal{F}_t$ and Bayes' rule (see LN Lemma 6.2.1 **2)** p. 105) yields

$$\mathbb{E}_{\mathbb{Q}}\left[f(N_{t+h} - N_t) \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[\frac{Z_{t+h}}{Z_t} f(N_{t+h} - N_t) \,\middle|\, \mathcal{F}_t\right] \tag{4}$$

$$= \mathbb{E}\left[\frac{Z_{t+h}}{Z_t} f(N_{t+h} - N_t)\right] \tag{5}$$

$$= \mathbb{E}\left[\frac{Z_{t+h}}{Z_t}\right] \mathbb{E}\left[f(N_{t+h} - N_t)\right] \tag{6}$$

$$= \mathbb{E}\left[f(N_{t+h} - N_t)\right], \tag{7}$$

since $\mathbb{E}\left[Z_{t+h}/Z_t\right] = \mathbb{E}\left[\mathbb{E}\left[Z_{t+h}/Z_t \,|\, \mathcal{F}_t\right]\right] = 1$. Because f was arbitrary, we conclude (by the hint) that $N_{t+h} - N_t$ and $\mathcal{F}_t$ are $\mathbb{Q}$-independent. For the Poisson distribution property, we take the Borel function $f(x) := \mathbb{1}_{\{x=k\}}$ and insert it in (7):

$$\mathbb{Q}\left[N_{t+h} - N_t = k\right] = \mathbb{E}\left[f(N_{t+h} - N_t)\right] = \mathbb{P}\left[N_{t+h} - N_t = k\right] = \frac{(\lambda h)^k}{k!} e^{-\lambda h},$$

i.e., $N_{t+h} - N_t$ is Poisson distributed with parameter λh.

For the SDE part, we note that by Girsanov's theorem (see LN Theorem 6.2.3 p. 106), $W^{\mathbb{Q}}$ is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion and that the SDE of S under $\mathbb{Q}$ is given by

$$\mathrm{d}S = S_-(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W + \kappa \,\mathrm{d}\widetilde{N}) = S_-(\sigma \,\mathrm{d}W^{\mathbb{Q}} + \kappa \,\mathrm{d}\widetilde{N}).$$

(c) By construction, $S = S^c S^d$, where S^c and S^d are from part a). Under $\mathbb{Q}$, we have (according to part a) and b))

$$S_T^c = e^{\sigma W_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T} \quad \text{and} \quad S_T^d = e^{\log(1+\kappa) N_T - \lambda \kappa T}.$$

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By $\mathbb{P}$-independence of W and N, we obtain the formula

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[S_T^\alpha\right] &= \mathbb{E}_{\mathbb{Q}}\left[(S_T^c)^\alpha (S_T^d)^\alpha\right] \\ &= \mathbb{E}\left[Z_T (S_T^c)^\alpha (S_T^d)^\alpha\right] \\ &= \mathbb{E}\left[Z_T (S_T^c)^\alpha\right] \mathbb{E}\left[(S_T^d)^\alpha\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[(S_T^c)^\alpha\right] \mathbb{E}\left[(S_T^d)^\alpha\right].\end{aligned}$$

It remains to compute the quantities $\mathbb{E}_{\mathbb{Q}}\left[(S_T^c)^\alpha\right]$ and $\mathbb{E}\left[(S_T^d)^\alpha\right]$ separately. We start with $\mathbb{E}\left[(S_T^d)^\alpha\right]$. The moment generating function of a Poisson random variable X with parameter λ is : $\phi_X(u) = \mathbb{E}_{\mathbb{P}}\left[e^{uX}\right] = e^{\lambda(e^u - 1)}$. We have

$$\mathbb{E}\left[(S_T^d)^\alpha\right] = e^{\lambda T((1+\kappa)^\alpha - 1) - \alpha\lambda\kappa T}.$$

Since $W_T^{\mathbb{Q}} \sim \mathcal{N}(0,T)$ under $\mathbb{Q}$, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[(S_T^c)^\alpha\right] = e^{\frac{1}{2}T\alpha\sigma^2(\alpha-1)}.$$

We finally obtain the formula

$$\mathbb{E}_{\mathbb{Q}}\left[S_T^\alpha\right] = e^{T\left(\frac{1}{2}\alpha\sigma^2(\alpha-1) + \lambda((1+\kappa)^\alpha - 1) - \alpha\kappa\lambda\right)}.$$

Exercise 14-2. Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the $\mathbb{P}$-null sets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0$ and the undiscounted stock price process $\widetilde{S}^1$ are given by $\widetilde{S}_t^0 = e^{rt}$ and $\widetilde{S}_t^1 = e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$, $0 \le t \le T$, $r, \mu \in \mathbb{R}$ and $\sigma > 0$. Denote by $\mathbb{Q}^*$ the unique equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$ on $\mathcal{F}_T$.

(a) Let $\widetilde{S}^2 = (\widetilde{S}_t^2)_{t\ge 0}$ be a strictly positive continuous semimartingale with respect to $\mathbb{P}$ and $\mathbb{F}$, which we interpret as the undiscounted price process of another traded asset. Let $\varphi_t = (\eta_t, \vartheta_t^2)$, $t \le 0 < T$, be a pair of adapted processes whose paths are continuous on $[0,T)$ for $\mathbb{P}$-almost all ω. Set $\widetilde{V}_t(\varphi) := \eta_t \widetilde{S}_t^0 + \vartheta_t^2 \widetilde{S}_t^2$ and suppose that $\widetilde{V}_t(\varphi) > 0$ $\mathbb{P}$-a.s. for all $0 \le t < T$. Define

$$\pi_t^0 := \frac{\eta_t \widetilde{S}_t^0}{\widetilde{V}_t(\varphi)} \quad \text{and} \quad \pi_t^2 := \frac{\vartheta_t^2 \widetilde{S}_t^2}{\widetilde{V}_t(\varphi)}, \quad 0 \le t < T.$$

Show that φ is *self-financing*, i.e. $\widetilde{V}_t(\varphi) = \widetilde{V}_0(\varphi) + \int_0^t \eta_s \,\mathrm{d}\widetilde{S}_s^0 + \int_0^t \vartheta_s^2 \,\mathrm{d}\widetilde{S}_s^2$ for all $0 \le t < T$ $\mathbb{P}$-a.s., if and only if we have $\mathbb{P}$-a.s for all $0 \le t < T$

$$\pi_t^0 + \pi_t^2 = 1 \quad \text{and} \quad \frac{\mathrm{d}\widetilde{V}_t(\varphi)}{\widetilde{V}_t(\varphi)} = \pi_t^0 \frac{\mathrm{d}\widetilde{S}_t^0}{\widetilde{S}_t^0} + \pi_t^2 \frac{\mathrm{d}\widetilde{S}_t^2}{\widetilde{S}_t^2}.$$

(b) Now assume that $\widetilde{S}^2$ denotes the undiscounted arbitrage-free price process of a European call option on $\widetilde{S}^1$ with strike $K = 1$ and maturity T. Recall that $\widetilde{S}_t^2 > 0$ $\mathbb{P}$-a.s. for all $0 \le t < T$ and satisfies $\mathbb{P}$-a.s. for all $0 \le t < T$

$$\begin{aligned}\mathrm{d}\widetilde{S}_t^2 &= \Phi(d_1)\,\mathrm{d}\widetilde{S}_t^1 - e^{-rT}\Phi(d_2)\,\mathrm{d}\widetilde{S}_t^0, \\ \widetilde{S}_t^2 &= \Phi(d_1)\widetilde{S}_t^1 - e^{-rT}\Phi(d_2)\widetilde{S}_t^0,\end{aligned}$$

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where $d_{1,2} = \frac{\log \widetilde{S}_t^1 + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and Φ denotes the cdf (distribution function) of a standard normal random variable. Derive a formula for the self-financing strategy $\varphi_t = (\eta_t, \vartheta_t^2)$, $t \leq 0 < T$, that replicates one stock $\widetilde{S}^1$ by trading only in $\widetilde{S}^0$ and $\widetilde{S}^2$.

Hint: Use part (a).

(c) Now assume that $\sigma = 1$. Prove that there exists a random variable X such that

$$\mathbb{E}_{\mathbb{Q}^*}[(S_t^1 - 1)^+] = \mathbb{Q}^*[X \leq t], \quad 0 \leq t \leq T,$$

and describe the law (distribution) of X under $\mathbb{Q}^*$.

Solution 14-2. (a) The first equation holds by definition for all $\varphi = (\eta, \vartheta^2)$ regardless of whether the strategy is self-financing or not. Next, note that $\widetilde{V}(\varphi)$ is adapted and has continuous paths on $[0,T)$ for $\mathbb{P}$-almost all ω, since the same is true for $\eta, \vartheta, \widetilde{S}^1, \widetilde{S}^2$. Since $\widetilde{V}(\varphi)$ is moreover strictly positive $\mathbb{P}$-a.s. for all $0 \leq t < T$, it follows that $\frac{1}{\widetilde{V}(\varphi)}$ is adapted and has continuous and strictly positive paths on $[0,T)$ for $\mathbb{P}$-almost all ω, too. In conclusion, both $\widetilde{V}(\varphi)$ and $\frac{1}{\widetilde{V}(\varphi)}$ are predictable and locally bounded on $[0,t]$ for all $t < T$. Hence by the associativity of the stochastic integral we have $\mathbb{P}$-a.s. for all $0 \leq t < T$

$$
\begin{aligned}
& \mathrm{d}\widetilde{V}_t(\varphi) = \eta_t \, \mathrm{d}\widetilde{S}_t^0 + \vartheta_t^2 \, \mathrm{d}\widetilde{S}_t^2 \\
\Leftrightarrow \quad & \mathrm{d}\widetilde{V}_t(\varphi) = \widetilde{S}_t^0 \eta_t \frac{\mathrm{d}\widetilde{S}_t^0}{\widetilde{S}_t^0} + \vartheta_t^2 \widetilde{S}_t^2 \frac{\mathrm{d}\widetilde{S}_t^2}{\widetilde{S}_t^2} \\
\Leftrightarrow \quad & \frac{\mathrm{d}\widetilde{V}_t(\varphi)}{\widetilde{V}_t(\varphi)} = \frac{\widetilde{S}_t^0 \eta_t}{\widetilde{V}_t(\varphi)} \frac{\mathrm{d}\widetilde{S}_t^0}{\widetilde{S}_t^0} + \frac{\vartheta_t^2 \widetilde{S}_t^2}{\widetilde{V}_t(\varphi)} \frac{\mathrm{d}\widetilde{S}_t^2}{\widetilde{S}_t^2} \\
\Leftrightarrow \quad & \frac{\mathrm{d}\widetilde{V}_t(\varphi)}{\widetilde{V}_t(\varphi)} = \pi_t^0 \frac{\mathrm{d}\widetilde{S}_t^0}{\widetilde{S}_t^0} + \pi_t^2 \frac{\mathrm{d}\widetilde{S}_t^2}{\widetilde{S}_t^2}, \qquad (8)
\end{aligned}
$$

which establishes the claim.

(b) Since $\widetilde{S}_t^2 > 0$ $\mathbb{P}$-a.s. for all $0 \leq t < T$, we have by part (a) $\mathbb{P}$-a.s. for all $0 \leq t < T$

$$\frac{\mathrm{d}\widetilde{S}_t^2}{\widetilde{S}_t^2} = \pi_t^0 \frac{\mathrm{d}\widetilde{S}_t^0}{\widetilde{S}_t^0} + \pi_t^1 \frac{\mathrm{d}\widetilde{S}_t^1}{\widetilde{S}_t^1}, \qquad (9)$$

where $\pi_t^0 = -\frac{e^{-rT}\Phi(d_2)\widetilde{S}_t^0}{\widetilde{S}_t^2}$ and $\pi_t^1 = \frac{\Phi(d_1)\widetilde{S}_t^1}{\widetilde{S}_t^2}$. Note that π^1 is adapted, strictly positive and continuous on $[0,T)$. Hence, the same is true for $\frac{1}{\pi^1}$, which is therefore predictable and locally bounded. By associativity of the stochastic integral, we may deduce that we have $\mathbb{P}$-a.s. for all $0 \leq t < T$

$$\frac{\mathrm{d}\widetilde{S}_t^1}{\widetilde{S}_t^1} = -\frac{\pi_t^0}{\pi_t^1} \frac{\mathrm{d}\widetilde{S}_t^0}{\widetilde{S}_t^0} + \frac{1}{\pi_t^1} \frac{\mathrm{d}\widetilde{S}_t^2}{\widetilde{S}_t^2}. \qquad (10)$$

Note that $-\frac{\pi_t^0}{\pi_t^1} + \frac{1}{\pi_t^1} = \frac{\pi^1}{\pi_t^1} = 1$. Now define $\varphi = (\eta, \vartheta^2)$ by

$$\eta := \frac{\widetilde{S}_t^1 \left(-\frac{\pi_t^0}{\pi_t^1}\right)}{\widetilde{S}_t^0} = e^{-rT} \frac{\Phi(d_2)}{\Phi(d_1)}, \qquad (11)$$

$$\vartheta^2 := \frac{\widetilde{S}_t^1 \frac{1}{\pi_t^1}}{\widetilde{S}_t^2} = \frac{1}{\Phi(d_1)}. \qquad (12)$$

It follows by part (a) that $\varphi = (\eta, \vartheta^2)$ is the desired self-financing strategy.

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(c) We know from the lecture that S is given by

$$S_t = e^{W_t^* - \frac{1}{2}t}, \quad 0 \le t \le T, \tag{13}$$

where $W^* = (W_t^*)_{t\ge 0}$ is a Brownian motion under $\mathbb{Q}^*$. Fix $t \in [0,T]$. Using that $W_t^* \sim \mathcal{N}(0,t)$ under $\mathbb{Q}^*$, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^*}[(S_t^1 - 1)^+] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \left(e^{-t/2+x} - 1\right)^+ e^{-\frac{x^2}{2t}}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi t}} \int_{t/2}^{\infty} \left(e^{-t/2+x} - 1\right) e^{-\frac{x^2}{2t}}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi t}} \int_{t/2}^{\infty} e^{-\frac{(x-t)^2}{2t}}\,\mathrm{d}x - \mathbb{Q}^*[W_t^* \ge t/2] \\
&= \mathbb{Q}^*[W_t^* \ge -t/2] - \mathbb{Q}^*[W_t^* \ge t/2] \\
&= \mathbb{Q}^*[-t/2 \le W_t^* \le t/2] = \mathbb{Q}^*[(W_t^*)^2 \le t^2/4] \\
&= \mathbb{Q}^*\left[\left(\frac{2W_t^*}{\sqrt{t}}\right)^2 \le t\right] = \mathbb{Q}^*[X \le t],
\end{aligned} \tag{14}$$

where $X = Y^2$ and $Y \sim \mathcal{N}(0, 2^2)$. Alternatively, we have $X = 4Z$, where $Z \sim \chi_1^2$.

Exercise 14-3. Fix a time horizon $T \in (0,\infty)$ and a probability space $(\Omega, \mathcal{F}, P)$ on which there is a Brownian motion $(W_t)_{0\le t\le T}$. We take as filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the one generated by W and augmented by the P-nullsets in $\sigma(W_s; s \le T)$. Consider the Black-Scholes model where the undiscounted bank account and the undiscounted risky asset price are given by

$$\frac{d\widetilde{S}_t^0}{\widetilde{S}_t^0} = r dt,$$

$$\frac{d\widetilde{S}_t^1}{\widetilde{S}_t^1} = \mu dt + \sigma dW_t,$$

where $\mu, r \in \mathbb{R}$ and $\sigma > 0$. We assume that $\widetilde{S}_0^0 = 1$ and $\widetilde{S}_0^1 > 0$.

(a) Consider the *n-th root of the stock* option, given by

$$\widetilde{H}_n = \left(\widetilde{S}_T^1\right)^{1/n},$$

for $n \in \{1, 2, \ldots\}$.

i) Compute the undiscounted arbitrage-free price $\widetilde{V}_t^{\widetilde{H}_n}$ at time t.
Hint: $E[e^{tX}] = e^{\frac{1}{2}\sigma^2 t^2}$ for $X \sim N(0, \sigma^2)$.

ii) Find the replicating strategy for $\widetilde{H}_n$.

(b) Let $\widetilde{H} = (\widetilde{S}_T^1 - 1)^+$ be a *call option*, and denote by $\widetilde{V}_t^{\widetilde{H}}$ its undiscounted arbitrage-free price at time t. Consider the option

$$\widetilde{J} = \begin{cases} \widetilde{S}_T^1 & \text{if } \widetilde{S}_T^1 < 1, \\ \left(\widetilde{S}_T^1\right)^2 & \text{if } \widetilde{S}_T^1 \ge 1, \end{cases}$$

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and denote $\widetilde{V}_t^{\widetilde{J}}$ its undiscounted arbitrage-free price at time t. Show that

$$\widetilde{V}_t^{\widetilde{J}} \geq e^{r(T-t)}\big(\widetilde{V}_t^{\widetilde{H}}\big)^2 + \widetilde{S}_t^1 + \widetilde{V}_t^{\widetilde{H}}.$$

Hint: Use Jensen's inequality.

Solution 14-3. (a) i) First, recall that

$$\widetilde{S}_t^1 = S_0^1 \exp^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t},$$

and that

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t$$

is a Brownian motion under the unique EMM. This will be used in the risk-neutral pricing formula:

$$\begin{aligned}
\widetilde{V}_t^{\widetilde{H}_n} &= e^{-r(T-t)}\mathbb{E}_Q\Big[\Big(\widetilde{S}_T^1\Big)^{1/n}|\mathcal{F}_t\Big] = e^{-r(T-t)}(\widetilde{S}_t^1)^{\frac{1}{n}}\mathbb{E}_Q\Big[\Big(\frac{\widetilde{S}_T^1}{\widetilde{S}_t^1}\Big)^{1/n}|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\Big(\widetilde{S}_t^1\Big)^{1/n}\mathbb{E}_Q\Big[\exp\Big(\frac{\sigma}{n}(W_T - W_t) + \frac{1}{n}(\mu - \frac{\sigma^2}{2})(T-t)\Big)|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\Big(\widetilde{S}_t^1\Big)^{1/n}\mathbb{E}_Q\Big[\exp\Big(\frac{\sigma}{n}(W_T^* - W_t^*) - \frac{\mu - r}{n}(T-t) + \frac{1}{n}(\mu - \frac{\sigma^2}{2})(T-t)\Big)|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\Big(\widetilde{S}_t^1\Big)^{1/n}\mathbb{E}_Q\Big[\exp\Big(\frac{\sigma}{n}(W_T^* - W_t^*) + \frac{1}{n}(r - \frac{\sigma^2}{2})(T-t)\Big)|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)+\frac{1}{n}(r-\frac{\sigma^2}{2})(T-t)}\Big(\widetilde{S}_t^1\Big)^{1/n}\mathbb{E}_Q\Big[\exp\Big(\frac{\sigma}{n}(W_T^* - W_t^*)\Big)|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)+\frac{1}{n}(r-\frac{\sigma^2}{2})(T-t)}e^{\frac{\sigma^2}{2n^2}(T-t)}\Big(\widetilde{S}_t^1\Big)^{1/n},
\end{aligned}$$

where in the last step we used the hint.

ii) We have,

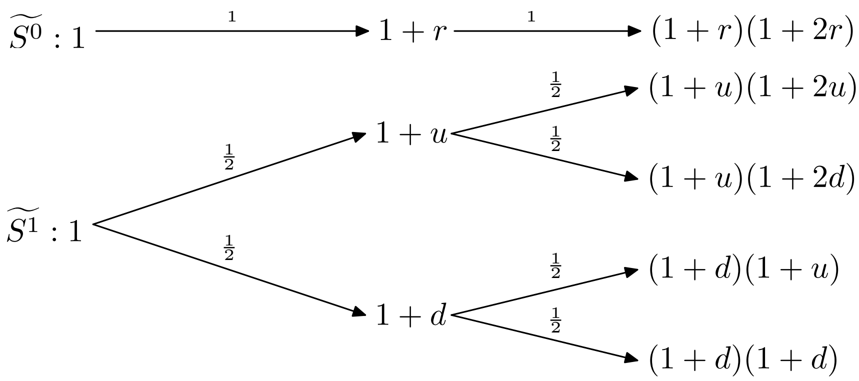
$$\begin{aligned}
\theta_t^{\widetilde{H}_n} &= \frac{\partial \widetilde{V}_t^{\widetilde{H}_n}}{\partial \widetilde{S}_t^1} = e^{-r(T-t)+\frac{1}{n}(r-\frac{\sigma^2}{2})(T-t)}e^{\frac{\sigma^2}{2n^2}(T-t)}\frac{1}{n}\Big(\widetilde{S}_t^1\Big)^{1/n-1}; \\
\eta_t^{\widetilde{H}_n} &= e^{-rt}\widetilde{V}_t^{\widetilde{H}_n} - e^{-rt}\theta_t^{\widetilde{H}_n}\widetilde{S}_t^1 = e^{-rT+\frac{1}{n}(r-\frac{\sigma^2}{2})(T-t)}e^{\frac{\sigma^2}{2n^2}(T-t)}\big(1 - \frac{1}{n}\big)\Big(\widetilde{S}_t^1\Big)^{1/n}.
\end{aligned}$$

(b) Just plugging in T and comparing both sides of the equation for the cases $\widetilde{S}_T^1 < 1$ and $\widetilde{S}_T^1 \geq 1$ gives that $\widetilde{J} = (\widetilde{H})^2 + \widetilde{S}_T^1 + \widetilde{H}$. Then

$$\begin{aligned}
\widetilde{V}_t^{\widetilde{J}} &= e^{-r(T-t)}\mathbb{E}_Q[\widetilde{J}|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}_Q[(\widetilde{H})^2 + \widetilde{S}_T^1 + \widetilde{H}|\mathcal{F}_t] \\
&\geq e^{-r(T-t)}\Big(\mathbb{E}_Q[\widetilde{H}|\mathcal{F}_t]\Big)^2 + \widetilde{S}_t^1 + \widetilde{V}_t^{\widetilde{H}} \\
&= e^{r(T-t)}\big(\widetilde{V}_t^{\widetilde{H}}\big)^2 + \widetilde{S}_T^1 + \widetilde{V}_t^{\widetilde{H}}
\end{aligned}$$

by Jensen's inequality.

Exercise 14-4. Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock. The movements of the bank account $\widetilde{S}^0$ and the stock price $\widetilde{S}^1$ are described by the following trees, where the numbers beside the branches denote transition probabilities and where $u > d$ and $d, r > -0.5$.



Note that the interest rate is $2r$ in the second period.

More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space with $\Omega := \{-1, 1\}^2$, $\mathcal{F} := 2^\Omega$ and the probability measure $\mathbb{P}$ defined by $\mathbb{P}[\{(x_1, x_2)\}] := p_{x_1} p_{x_1, x_2}$, where

$$p_1 = p_{-1} := \frac{1}{2} \quad \text{and} \quad p_{1,1} = p_{1,-1} = p_{-1,1} = p_{-1,-1} := \frac{1}{2}.$$

Next, consider Y_1 and Y_2 given by

$$\begin{aligned} Y_1((1,1)) &:= Y_1((1,-1)) := 1 + u, & Y_1((-1,1)) &:= Y_1((-1,-1)) := 1 + d, \\ Y_2((1,1)) &:= 1 + 2u, & Y_2((-1,1)) &:= 1 + u, \\ Y_2((1,-1)) &:= 1 + 2d, & Y_2((-1,-1)) &:= 1 + d. \end{aligned}$$

The bank account process $\widetilde{S}^0$ and the stock price process $\widetilde{S}^1$ are then given by $\widetilde{S}^0_k = \prod_{j=1}^k (1 + jr)$ and $\widetilde{S}^1_k = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$, respectively. Finally, the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ is defined by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1)$ and $\mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F}$.

(a) Prove in detail that the market $(\widetilde{S}^0, \widetilde{S}^1)$ is free of arbitrage if and only if both $d < r < u$ and $d < 2r < u$ are satisfied.

(b) Suppose that $u = 0.02$, $r = 0.01$ and $d = -0.01$. Give an example of a self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ satisfying $\mathbb{P}[V_2(\varphi) \geq 1000] = 0.25$ and $V_2(\varphi) \geq 0$ $\mathbb{P}$-a.s.

(c) Suppose again that $u = 0.02$, $r = 0.01$ and $d = -0.01$. Does there exist a self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ satisfying $V_2(\varphi) \geq 1000$ $\mathbb{P}$-a.s.? Justify your answer by either providing a concrete example of such a strategy or by formally arguing that such a strategy does not exist.

Solution 14-4. (a) By the fundamental theorem of asset pricing in discrete time (Theorem 2.2.1 in the lecture notes), the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free if and only if there exists an equivalent martingale measure (EMM) $\mathbb{Q}$ for the discounted stock price process S^1.

Any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_2$ can be described by

$$\mathbb{Q}[\{(x_1, x_2)\}] := q_{x_1} q_{x_1, x_2},$$

where $q_{x_1}, q_{x_1,x_2} \in (0,1)$ satisfying $\sum_{x_1\in\{-1,1\}} q_{x_1} = 1$ and $\sum_{x_2\in\{-1,1\}} q_{x_1,x_2} = 1$ for all $x_1 \in \{-1,1\}$. Next, since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and Y_1 only takes two values, S^1 is a $\mathbb{Q}$-martingale if and only if $q_1, q_{1,1}, q_{-1,1} \in (0,1)$ and

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{Y_1}{1+r}\right] = 1 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_2}{1+2r}\,\middle|\, Y_1 = (1+u)\right] = 1$$
$$\text{and} \quad \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_2}{1+2r}\,\middle|\, Y_1 = (1+d)\right] = 1. \tag{15}$$

This is equivalent to $q_1, q_{1,1}, q_{-1,1} \in (0,1)$ and

$$\begin{aligned}
q_1 \times (1+u) + (1-q_1)\times(1+d) = 1+r \quad &\iff \quad q_1 = \frac{r-d}{u-d},\\
q_{1,1}\times(1+2u) + (1-q_{1,1})\times(1+2d) = 1+2r \quad &\iff \quad q_{1,1} = \frac{2r-2d}{2u-2d},\\
q_{-1,1}\times(1+u) + (1-q_{-1,1})\times(1+d) = 1+2r \quad &\iff \quad q_{-1,1} = \frac{2r-d}{u-d}.
\end{aligned} \tag{16}$$

In conclusion, the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free if and only if

$$\frac{r-d}{u-d} \in (0,1) \quad \text{and} \quad \frac{2r-d}{u-d} \in (0,1) \quad \iff \quad d<r<u \quad \text{and} \quad d<2r<u. \tag{17}$$

(b) Note that we have $u = 2r$, so the market is not free of arbitrage by part (a). The idea is to short the stock in the case of an “down-movement in the first period. To this end, consider the strategy $\varphi \,\widehat{=}\, (0,\vartheta)$, where

$$\vartheta_1^1 := 0, \quad \vartheta_2^1((1,1)) := \theta_2^1((1,-1)) := 0, \quad \vartheta_2^1((-1,1)) := \vartheta_2^1((-1,-1)) := -c, \tag{18}$$

where $c > 0$ is to be determined. Then ϑ is predictable and we have

$$\begin{aligned}
V_2(\varphi)((1,1)) &= 0 + 0\times \Delta S_1^1((1,1)) + 0 \times \Delta S_2^1((1,1)) = 0,\\
V_2(\varphi)((1,-1)) &= 0 + 0\times \Delta S_1^1((1,-1)) + 0 \times \Delta S_2^1((1,-1)) = 0,\\
V_2(\varphi)((-1,1)) &= 0 + 0\times \Delta S_1^1((-1,1)) - c \times \Delta S_2^1((-1,1))\\
&= -c \times \left(\frac{(1+d)(1+2r)}{(1+r)(1+2r)} - \frac{1+d}{1+r}\right) = -c\times 0 = 0,\\
V_2(\varphi)((-1,-1)) &= 0 + 0\times \Delta S_1^1((-1,-1)) - c \times \Delta S_2^1((-1,-1))\\
&= -c \times \left(\frac{(1+d)(1+d)}{(1+r)(1+2r)} - \frac{1+d}{1+r}\right)\\
&= -c \times \left(\frac{1+d}{1+r} \times \frac{d-2r}{1+2r}\right) = c \times \frac{0.99\times 0.03}{1.01\times 1.02}.
\end{aligned} \tag{19}$$

Choosing c large enough, i.e. $c \geq 1000 \times \frac{1.01\times1.02}{0.99\times0.03} = 34686.86$ gives the desired strategy as $\mathbb{P}[\{(-1,1)\}] = 1/2\times 1/2 = 0.25$.

(c) Such a strategy does **not** exist. Seeking a contradiction, suppose that there exists a strategy $\varphi \,\widehat{=}\, (0,\vartheta)$ such that $V_2(\varphi) \geq 1000$ $\mathbb{P}$-a.s. Then in particular we have $V_2(\varphi)((-1,1)) \geq 1000$. Since $\Delta S_2^1((-1,1)) = 0$ (see above), it follows that $V_1(\varphi)((-1,1)) \geq 1000$. But given that $d < r < u$, the market $(\widetilde{S}^0, \widetilde{S}^1)$ is free of arbitrage in the first-period and since $V_0(\varphi) = 0$, we necessarily have $V_1(\varphi)((1,1)) = V_1(\varphi)((1,-1)) < 0$. Again since $d < r < u$, after an up-movement in the first period the market $(\widetilde{S}^0, \widetilde{S}^1)$ is free of arbitrage in the second period. Thus we cannot have $V_1(\varphi)((1,1)) = V_1(\varphi)((1,-1)) < 0$ and $V_2(\varphi)((1,1)) \geq 1000 > 0$ and $V_2(\varphi)((1,-1)) \geq 1000 > 0$. Thus, we arrive at a contradiction.

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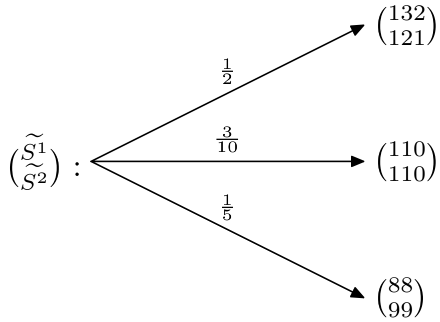
Exercise 14-5. Consider a one-period financial market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ consisting of a bank account $\widetilde{S}^0$ with interest rate $r := 0.1$ and two stocks $\widetilde{S}^1, \widetilde{S}^2$. The movements of $\widetilde{S}^1$ and $\widetilde{S}^2$ are given by the following trees, where the numbers beside the branches denote transition probabilities.



More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space with $\Omega := \{1, 0, -1\}$, $\mathcal{F} := 2^\Omega$ and the probability measure $\mathbb{P}$ defined by $\mathbb{P}[\{1\}] := 0.5$, $\mathbb{P}[\{0\}] := 0.3$ and $\mathbb{P}[\{-1\}] := 0.2$. Next, consider Y_1^1 and Y_2^1 given by

$$Y_1^1(1) = 1.32, \qquad Y_1^1(0) := 1.1, \qquad Y_1^1(-1) := 0.88,$$
$$Y_1^2(1) = 1.21, \qquad Y_1^2(0) := 1.1, \qquad Y_1^2(-1) := 0.99,$$

The movements of the bank account $\widetilde{S}^0$ and the two stocks $\widetilde{S}^1$ and $\widetilde{S}^2$ are then given by

$$\widetilde{S}_0^0 := 1, \quad \widetilde{S}_0^1 := \widetilde{S}_0^2 := 100, \quad \widetilde{S}_1^0 := 1.1, \quad \widetilde{S}_1^1 := 100Y_1^1, \quad \widetilde{S}_1^2 := 100Y_1^2.$$

Finally, the filtration $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$ is defined by $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_1 := 2^\Omega = \mathcal{F}$.

(a) Show that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is free of arbitrage and incomplete.

(b) The *undiscounted* payoff of an *exchange option* is given by

$$\widetilde{H}^{EX} := \left(\widetilde{S}_1^1 - \widetilde{S}_1^2\right)^+ := \max\left(0, \widetilde{S}_1^1 - \widetilde{S}_1^2\right).$$

Compute the set of all arbitrage-free prices for $\widetilde{H}^{EX}$. Does there exist an admissible self-financing strategy $\varphi \mathrel{\hat{=}} (3, \vartheta)$ such that $V_1(\varphi) = \frac{\widetilde{H}^{EX}}{1+r}$ $\mathbb{P}$-a.s.?

(c) Compute an admissible self-financing strategy $\varphi \mathrel{\hat{=}} (5, \vartheta)$, which *superreplicates* $\widetilde{H}^{EX}$, i.e. satisfies $V_1(\varphi) \geq \frac{\widetilde{H}^{EX}}{1+r}$ $\mathbb{P}$-a.s.

Solution 14-5. (a) By the fundamental theorem of asset pricing in discrete time (Theorem 2.2.1 in the lecture notes), showing that the market is arbitrage-free is equivalent to showing that there exists an equivalent martingale measure (EMM) $\mathbb{Q}$ for the discounted stock prices $S = (S^1, S^2)$. Note that the movements of the discounted stock price processes S^1 and S^2 are given by the following trees, where the numbers beside the branches denote transition probabilities.

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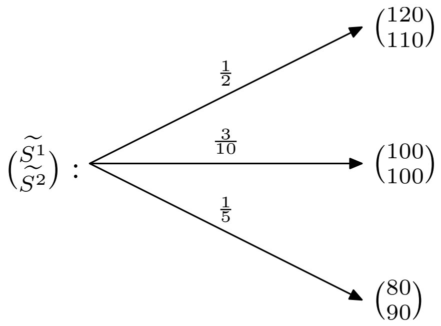

Observe that $S^2 = \frac{1}{2}S^1 + 50$. Hence $S = (S^1, S^2)$ is a $\mathbb{Q}$-martingale if and only if S^1 is a $\mathbb{Q}$-martingale. Next, any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_1$ can be described by a probability vector (q_1, q_0, q_{-1}), where $q_1 := \mathbb{Q}[\{1\}]$, $q_0 := \mathbb{Q}[\{0\}]$, $q_{-1} := \mathbb{Q}[\{-1\}]$ and $0 < q_1, q_0, q_{-1} < 1$. Then S^1 and hence S is a $\mathbb{Q}$-martingale if and only if

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[S_1^1] &= S_0^1, \\ 0 < q_1, q_0, q_{-1} &< 1. \end{aligned} \tag{20}$$

This is equivalent to

$$\begin{aligned} 120 \times q_1 + 100 \times q_0 + 80 \times q_{-1} &= 100, \\ q_1 + q_0 + q_{-1} &= 1, \\ 0 < q_1, q_0, q_{-1} &< 1, \end{aligned} \tag{21}$$

which is equivalent to

$$\begin{aligned} 20 \times q_1 - 20q_{-1} &= 0, \\ q_1 + q_0 + q_{-1} &= 1, \\ 0 < q_1, q_0, q_{-1} &< 1, \end{aligned} \tag{22}$$

which is in turn equivalent to

$$\begin{aligned} q_1 &= q_{-1} \\ q_0 &= 1 - 2q_1, \\ 0 < q_1, q_0, q_{-1} &< 1. \end{aligned} \tag{23}$$

Thus, the set $\mathbb{P}_e(S)$ of all equivalent martingale measures for S can be described by

$$\mathbb{P}_e(S) = \{(\lambda, 1 - 2\lambda, \lambda) \mid \lambda \in (0, 0.5)\}. \tag{24}$$

Since $\mathbb{P}_e(S)$ is nonempty and consist of more than one element, the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is arbitrage-free and incomplete.

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(b) Denote by $\mathbb{Q}_\lambda$ the EMM corresponding to the probability vector $(\lambda, 1-2\lambda, \lambda)$. Then the set $\mathcal{P}_{\widetilde{H}^{EX}}$ of all arbitrage-free prices for $\widetilde{H}^{EX}$ is given by

$$\begin{aligned} \mathcal{P}_{\widetilde{H}^{EX}} &= \Big\{ \mathbb{E}_{\mathbb{Q}_\lambda}\Big[\frac{\widetilde{H}^{EX}}{1+r}\Big] \,\Big|\, \lambda \in (0, 0.5) \Big\} \\ &= \{ \lambda \times 10 + (1-2\lambda) \times 0 + \lambda \times 0 \mid \lambda \in (0, 0.5) \} \\ &= (0, 5)\,. \end{aligned} \tag{25}$$

The set $\mathcal{P}_{\widetilde{H}^{EX}}$ is an nonempty open interval. In particular, the mapping $\mathbb{P}_e(S) \to \mathbb{R}$, $\mathbb{Q} \mapsto \mathbb{E}_{\mathbb{Q}}[\frac{\widetilde{H}^{EX}}{1+r}]$ is not constant. By the characterisation of attainable payoffs (Theorem 3.1.2 in the lecture notes) it follows immediately that $\widetilde{H}^{EX}$ is not attainable. Hence, there does not exist an admissible self-financing strategy $\varphi \,\hat{=}\, (3, \vartheta)$ with $V_1(\varphi) = \frac{\widetilde{H}^{EX}}{1+r}$ $\mathbb{P}$-a.s.

(c) Using that $S^2 = \frac{1}{2}S^1 + 50$, we may assume without loss of generality that $\vartheta^2 \equiv 0$, i.e. we only use the bank account and the first stock for hedging. Hence consider a self-financing strategy $\varphi \,\hat{=}\, (5, \vartheta)$, with $\vartheta_1^1 = c$ and $\vartheta_1^2 = 0$, where $c \in \mathbb{R}$ is to be determined. Then φ is a superreplication strategy for $\widetilde{H}^{EX}$ if and only if

$$\begin{aligned} 5 + c \times \Delta S_1^1(1) \geq \frac{\widetilde{H}^{EX}(1)}{1+r} \quad &\Longleftrightarrow \quad 5 + c \times 20 \geq 10 \quad \Longleftrightarrow \quad c \geq 1/4, \\ 5 + c \times \Delta S_1^1(0) \geq \frac{\widetilde{H}^{EX}(0)}{1+r} \quad &\Longleftrightarrow \quad 5 + c \times 0 \geq 0 \quad \Longleftrightarrow \quad c \in \mathbb{R}, \\ 5 + c \times \Delta S_1^1(-1) \geq \frac{\widetilde{H}^{EX}(-1)}{1+r} \quad &\Longleftrightarrow \quad 5 - c \times 20 \geq 0 \quad \Longleftrightarrow \quad c \leq 1/4. \end{aligned} \tag{26}$$

Choosing $c = 1/4$ gives the desired superreplication strategy.