Problem Sheet 1

Problem 1.   Convergence rate of the derivative

(1a)   (Exercise 1.4, Question 2 on p. 14 in the book "A First Course in Numerical Methods")
Carry out derivation and calculations (analogous to those in Example 1.2 on p. 6 in the book),
using the expression

$$\frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

for approximating the first derivative $f'(x_0)$. Show that the discretization error is $\mathcal{O}(h^2)$. More
precisely, the leading term of the error is $-\frac{h^2}{6} f'''(x_0)$ (wrong in the book) if $f'''(x_0) \neq 0$.

(1b)   (Exercise 1.4, Question 3 on p. 15 in the book "A First Course in Numerical Methods")
Carry out similar Matlab calculations to those in Example 1.3 (estimate $f'(x)$ with $f(x) = \sin(x)$,
$x_0 = 1.2$) using the approximation from (1a). Observe similarities and differences by comparing
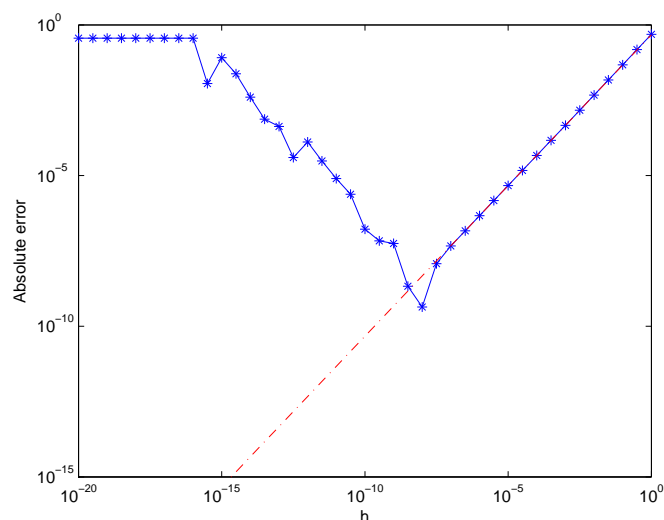your graph against that in Figure 1 below (Figure 1.3 in the book).

Figure 1: The combined effect of discretization and roundoff errors for a $\mathcal{O}(h)$ method

Problem 2.   Truncation and roundoff error

(Exercise 2.5, Question 13 on p. 34 in the book "A First Course in Numerical Methods") Consider the approximation to the first derivative

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

The truncation (or discretization) error for this formula is $\mathcal{O}(h)$. Suppose that the absolute error in evaluating the function f is bounded by ϵ and let us ignore the error generated in basic arithmetic operations.

(2a)   Show that the total computational error (truncation and rounding combined) is bounded by

$$\frac{Mh}{2} + \frac{2\epsilon}{h},$$

where M is a bound on $|f''(x)|$.

(2b)   What is the value of h for which the bound in (2a) is minimized?

(2c)   The rounding unit ϵ we employ is approximately equal to 10^{-16}. Use this to explain the behavior of the graph in Figure 1 (Figure 1.3 in the book). Make sure to explain the shape of the graph as well as the value where the apparent minimum is attained.

Problem 3.   Rounding errors in the quadratic equation

A quadratic equation $ax^2 + bx + c = 0 \quad a, b, c \in \mathbb{R}, \ a \neq 0$ can be solved by the quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Rounding errors however play a role when doing so.

(3a)   Implement a solver for the quadratic equation using the above formula in the Matlab function $\texttt{x = quadraticSolve(a,b,c)}$, which returns the two solutions x_1 and x_2 in the vector x.

(3b)   Compute a reference solution of the quadratic equation using the Matlab function $\texttt{roots}$. Compare your solutions of (3a) with this reference solutions, and compute the difference (for each solution x_1 and x_2 in the vector x separately). Implement this in the Matlab function $\texttt{compRoundErr(a,b,c)}$.

The error should be small for the pair $(a, b, c) = (1, 10^8, 10^6)$ and large for the pairs $(a, b, c) = (1, 10^8, \pi)$ and $(a, b, c) = (1, -10^8, \pi)$. Explain why, and point to the source of the error.

HINT: To get the two solutions x_1 and x_2 you may use for instance $\texttt{x1=max(x)}$ and $\texttt{x2=min(x)}$.

(3c)   A relation between the two solutions due to Vieta is known,

$$x_1 x_2 = \frac{c}{a}.$$

Use this to modify the algorithm to compute the two solutions of the quadratic equation, such that the rounding errors are small. Implement this new algorithm in the Matlab function $\texttt{x = quadraticSolve2(a,b,c)}$. Measure the relative error of this new method, by modifying the function $\texttt{compRoundErr(a,b,c)}$ from (3b) accordingly.

HINT: The large errors due to cancellation only appears in one of the two solutions.

Issue date: Sep. 19th, 2014.

To be solved by Sept. 29th, 2014 (for exercise groups on Monday) or Oct. 2nd, 2014 (for exercise groups on Thursday).