SCP-LECTURES

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Contents

Set

$$S_1 := \{a^2 + b^2 + 10c^2 : a, b, c \in \mathbf{Z}\},$$
$$S_2 := \{2a^2 + 2b^2 + 3c^2 - 2ac : a, b, c \in \mathbf{Z}\}.$$

It follows from local considerations that both S_1 and S_2 are contained in the subset

$$\mathbf{Z}' := \mathbf{Z}_{\geq 0} - \{4^a(16k + 6) : a \in \mathbf{Z}_{\geq 0}, k \in \mathbf{Z}\}$$

of the integers. It's been known for awhile (since Hasse, at least) that

$$\mathbf{Z}' = S_1 \cup S_2,$$

but it's a relatively recent theorem (about 25 years ago, due to Duke–Schulze-Pillot building off a technical breakthrough of Iwaniec) that there exists $X > 0$ such that

$$(X, \infty) \cap \mathbf{Z}' \subset S_1.$$

An aim of the course is to attempt to present a "modern" proof of this, largely as an excuse to develop the required machinery. In the first lecture, we also considered the problem of establishing upper bounds for $n(p) := \min\{a \in \mathbf{N} : (a|p) = -1\}$ and briefly discussed the QUE problem. We mentioned that these problems all turn out to be closely related to the *subconvexity problem* (without defining the latter). The initial aim of the course is to develop some background that should help both in motivating that problem and (later) in studying it.

The following general references that may be of use: [4] (especially Ch. 5), [2], [5], [6].

Some of the topics covered in these notes were not covered in the lectures but are included for the sake of completeness.

We now turn to a more detailed overview of the contents of the sections that follow. In §1 we review some basic properties of Fourier analysis on the multiplicative group of reals $\mathbf{R}^\times$, with an emphasis on the relation between asymptotic expansions of a function $f : \mathbf{R}^\times \to \mathbf{C}$ near the "boundary points" $\infty, 0$ and the meromorphic continuation and description of poles of its Mellin transform $f^\wedge(\chi)$; here $\chi(y) = |y|^s (y/|y|)^a$, $s \in \mathbf{C}, a \in \{0, 1\}$ is a character of $\mathbf{R}^\times$. This relation is presented first in a simplified context with $\mathbf{R}^\times$ replaced by the group of integers $\mathbf{Z}$. Besides recalling the above relationship, we define a certain *regularized Mellin transform* which has the effect of making precise a sense in which one can formally discard divergent integrals such as

$$\int_{y \in \mathbf{R}_+^\times} y^s \, d^\times y := 0.$$

Everything we do makes sense with $\mathbf{R}$ replaced by a general local field F; we restrict to the case $F = \mathbf{R}$ for the sake of focus. We refer to [3] for more discussion on using Mellin transforms to study asymptotics of sums. Our discussion of the regularized Mellin transform is based on the recommended article [9].

In §2 we describe Tate's local functional equation over $\mathbf{R}$, which compares the Mellin transforms of the restriction to $\mathbf{R}^\times$ of a Schwartz function f on $\mathbf{R}$ and that of its Fourier transform $\mathcal{F}f$; alternatively, it may be viewed as a computation of the Fourier transform of the tempered distributions $\chi(y) \, d^\times y$ on $\mathbf{R}$ for χ a character of $\mathbf{R}^\times$. Once again, the contents of this section make sense over a general local field, and indeed their generality is a key virtue; we restrict to the case of the reals for the sake of focus and diversity of presentation. We also include a brief discussion of Dirichlet characters, Gauss sums, and the "local functional equation over $\mathbf{Z}/q$." For more on the local functional equation discussed in §2 we refer to Tate's thesis (available in Cassels–Fröhlich's *Algebraic Number Theory*), [7], or [1, §3.1]; we also present Weil's treatment [8] in §2.4.

In §3 we talk a bit about Dirichlet series and how their meromorphic continuation and polar behavior relates to the asymptotic expansion of smoothly weighted sums of their coefficients. We talk some about functional equations, approximate functional equations, and motivate the notion of an analytic conductor of a Dirichlet series. We have for the most part eschewed stating precise theorems in this section because the hypotheses vary substantially between applications, and it seems better to learn the general principles and to apply them anew each time.

In §4 we establish the basic analytic properties of the Riemann zeta function $\zeta(s)$ and the Dirichlet L-functions $L(\chi, s)$. Our treatment follows something resembling

Tate's approach but (for the sake of variety of presentation) without the adelic language. References for this section include [2], [4], Tate's thesis, [7], etc.

1. Review on the Fourier/Mellin transforms

1.1. **The Fourier transform on Z.**

1.1.1. *Definition and basic properties.* For $r > 0$, write

$$\mathbf{C}^{(r)} := \{z \in \mathbf{C} : |z| = r\}.$$

Let $f : \mathbf{Z} \to \mathbf{C}$ be a function, which we suppose for now to be absolutely summable. Define its Fourier transform $f^\wedge$, whose domain we take to be the unit circle $\mathbf{C}^{(1)}$, by the absolutely convergent formula

$$f^\wedge(z) := \sum_n f(n) z^{-n}. \tag{1}$$

This is sometimes also referred to as the generating function attached to f. If $f^\wedge$ is integrable on $\mathbf{C}^{(1)}$, then the Fourier inversion theorem implies that we may recover f by the formula

$$f(n) = \int_{\mathbf{C}^{(1)}} f^\wedge(z) z^n \frac{dz}{2\pi i z}.$$

Here and always the contour is taken with the positive (counter-clockwise) orientation. The relationship between f and $f^\wedge$ satisfies the following formal identities (for $\zeta \in \mathbf{C}^{(1)}$, $b \in \mathbf{Z} - \{0\}$, $c \in \mathbf{Z}$, $a_k \in \mathbf{C}$, $c_k \in \mathbf{Z}$), the first of which is perhaps the most important.

(2)

$f(n)$	$f^\wedge(z)$
$f(n+c)$	$z^c f^\wedge(z)$
$\zeta^n f(n)$	$f^\wedge(z/\zeta)$
$n f(n)$	$-\frac{d}{dz} f^\wedge(z)$
$f(bn)$	$\|b\|^{-1} \sum_{\omega : \omega^b = z} f^\wedge(\omega)$
$f(-n)$	$f^\wedge(z^{-1})$
$\sum_{k=1}^K a_k f(c_k - n)$	$(\sum_{k=1}^K a_k z^{c_k}) f^\wedge(z^{-1})$
$f(n+1) - f(n)$	$(z-1) f^\wedge(s)$
$\sum_{m=-\infty}^{n-1} f(m)$	$(z-1)^{-1} f^\wedge(s)$

One can use the above transforms and formal identities to relate properties of f and $f^\wedge$. For example, one sees by iterating the third identity that $f^\wedge$ is smooth if and only if $f(n) \ll (1 + |n|)^{-A}$ for each fixed $A \geq 1$.

The assumption that f be absolutely summable can and shall be modified slightly. For a general function $f : \mathbf{Z} \to \mathbf{C}$, the set of r for which

$$\sum_{n \in \mathbf{Z}} r^{-n} |f(n)| < \infty$$

contains a maximal open interval which we denote by $(r_{+\infty}, r_{-\infty}) \subset (0, \infty)$ and call the *fundamental interval* of f. The notation is motivated by observing that as long as the fundamental interval is nonempty, the endpoint $r_{+\infty}$ is the infimum of all $r > 0$ for which

$$\sum_{n \in \mathbf{Z}_{\geq 0}} r^{-n} |f(n)| < \infty$$

and $r_{-\infty}$ is likewise the supremum of all $r > 0$ for which

$$\sum_{n \in \mathbf{Z}_{\leq 0}} r^{-n} |f(n)| < \infty.$$

For example, if $f(n) \asymp a_{\pm\infty}^n$ for $n \to \pm\infty$ and some fixed $0 < a_{+\infty} < a_{-\infty}$, then the fundamental interval is $(a_{+\infty}, a_{-\infty})$.

Example 1.1. The function $f(n) = e^{-e^n}$ decays superexponentially at $+\infty$ and behaves like 1 near $-\infty$, hence has fundamental interval $(0, 1)$. The function $f(n) = e^{-e^n} - 1$ behaves like 1 near $+\infty$ and like e^n near $-\infty$, and so has fundamental interval $(1, e)$. The function $f(n) = e^{-e^n} - 1 + e^n$ has fundamental interval (e, e^2) because it behaves like e^n near $+\infty$ and like $\frac{1}{2}e^{2n}$ near $-\infty$.

The set of complex numbers $z \in \mathbf{C}$ with $r_0 < |z| < r_1$ will be called the *fundamental annulus* of f. The formula (1) defines $f^\wedge$ as a holomorphic function on the fundamental annulus. If $f^\wedge$ if absolutely integrable on some circle $\mathbf{C}^{(r)}$ with $r \in (r_0, r_1)$, it satisfies the inversion formula

$$f(n) = \int_{\mathbf{C}^{(r)}} f^\wedge(z) z^n \, \frac{dz}{2\pi i z}. \tag{3}$$

We shall still refer to $f^\wedge$ as the Fourier transform of f, even if it may not be not defined on $\mathbf{C}^{(1)}$. However, we caution that it is important to keep in mind not only the function $f^\wedge$ but also its fundamental interval or annulus, since (for instance) the distinct functions f arising in Example 1.1 have the property that their Fourier transforms $f^\wedge$ admit coincident meromorphic continuations (but with disjoint fundamental intervals).

1.1.2. *Asymptotic expansions and meromorphic continuation.* It may happen that the function $f^\wedge$ admits an analytic continuation to a larger region than the fundamental annulus of f. This possibility roughly corresponds to the existence of asymptotic expansions of f near the boundary of $\mathbf{Z}$.

For example, suppose f has fundamental interval $(r_{+\infty}, r_{-\infty})$ and that for some larger interval $(R_{+\infty}, R_{-\infty})$ the function $f^\wedge$ extends meromorphically to some open set containing the annulus $R_{+\infty} \leq |z| \leq R_{-\infty}$. Assume moreover that $f^\wedge$ has finitely many poles in that region none of which lie on the circles $|z| = R_{\pm\infty}$. Then by shifting the contour in (3) to the circle $\mathbf{C}^{(R_{\pm\infty})}$ and estimating the remaining integral trivially, we see that

$$f(n) = f_{\pm\infty}(n) + O(R_{\pm\infty}^n) \text{ as } n \to \pm\infty \tag{4}$$

where $f_{\pm\infty}$ is the function obtained by summing, for each polar part

$$\pm\, c \frac{b! a}{(z - a)^{b+1}} \quad (c \in \mathbf{C}, b \in \mathbf{Z}_{\geq 0}, a \in \mathbf{C}^\times) \tag{5}$$

of $f^\wedge$ encountered in the outer annulus $r_{+\infty} < |z| < R_{+\infty}$ if the sign is $+$ or in the inner annulus $R_{-\infty} < |z| < r_{-\infty}$ if the sign is $-$, the function

$$n \mapsto c a^n n^b. \tag{6}$$

Exercise 1.2. Show in this way (i.e., without using the well-known explicit formula) that the Fibonacci sequence $f : \mathbf{Z} \to \mathbf{C}$, defined by $f(n) := 0$ for $n < 0$, $f(0) := f(1) := 1$, and $f(n) := f(n-1) + f(n-2)$ for $n \geq 2$, satisfies $f(n) \sim \varphi^n$ for $n \to +\infty$ with $\varphi := \frac{1+\sqrt{5}}{2}$. (This exercise is only meant to be illustrative; it is overkill to use Cauchy's theorem in an example like this where $f^\wedge$ turns out to be a rational function.)

Conversely, suppose f admits asymptotic expansions (4) for some interval $(R_{+\infty}, R_{-\infty})$ containing the fundamental interval and some finite linear combinations $f_{\pm\infty}$ of functions as in (6). Then $f^\wedge$, defined initially as a holomorphic function on the fundamental annulus, admits a meromorphic continuation to the annulus $R_{+\infty} < |z| < R_{-\infty}$ which is holomorphic away from the polar parts (5) corresponding to $f_{\pm\infty}$ as above. To see this, one splits the sum in the definition (1) into the subsums over $n \geq 0$ and $n < 0$, subtracting $f_{+\infty}$ off from the former and $f_{-\infty}$ off from the latter, giving for z in the fundamental annulus the identity

$$f^\wedge(z) = \sum_{n=1}^{\infty}(f(n) - f_{+\infty}(n))z^{-n} + \sum_{n=-\infty}^{0}(f(n) - f_{-\infty}(n))z^{-n} + R(n), \tag{7}$$

where

$$R(n) := \sum_{n=1}^{\infty} f_{+\infty}(n)z^{-n} + \sum_{n=-\infty}^{0} f_{-\infty}(n)z^{-n}. \tag{8}$$

The first two terms on the RHS of (7) extend holomorphically to $R_{+\infty} < |z| < R_{-\infty}$ thanks to (4), while $R(n)$ is an explicit meromorphic (in fact, rational) function of z on all of $\mathbf{C}^\times$ with the claimed polar parts as in (5), as follows for instance from the computations

$$\sum_{n=1}^{\infty} a^n n^b z^{-n} = \frac{b!a}{(z-a)^{b+1}}, \qquad \sum_{n=-\infty}^{0} a^n n^b z^{-n} = \frac{-b!a}{(z-a)^{b+1}} \tag{9}$$

valid respectively for $|z| > a$, $|z| < a$. Therefore (7) gives the desired meromorphic continuation of $f^\wedge$.

1.1.3. *Finite functions.* We pause to put a key concept from the preceeding section in a more general context.

Definition 1.3. Let G be an abelian group. Given $f : G \to \mathbf{C}$ and $h \in G$, define $f_h : G \to \mathbf{C}$ by the formula $f_h(g) := f(gh)$. Say that f is a *finite function* if the span of the functions f_h $(h \in G)$ is finite-dimensional.

Exercise 1.4. Suppose G is infinite. For each finite collection $(f_i)_{i \in I}$ of finite functions $f_i : G \to \mathbf{C}$ there exists a natural number K and distinct group elements $c_1, \dots, c_K \in G$ and nonzero complex coefficients $a_1, \dots, a_K$ so that $\sum_{k=1}^{K} a_k f_i(gc_k) = 0$ for each $i \in I$, $g \in G$.

Exercise 1.5. Finite functions on $\mathbf{Z}$ are finite linear combinations of $n \mapsto a^n n^b$ for $a \in \mathbf{C}^\times$, $b \in \mathbf{Z}_{\geq 0}$. Finite functions on $\mathbf{R}$ are finite linear combinations of $x \mapsto e^{ax}x^b$ for $a \in \mathbf{C}$, $b \in \mathbf{Z}_{\geq 0}$. Finite functions on $\mathbf{R}_+^\times$ are finite linear combinations of $y \mapsto y^a \log(y)^b$ for $a \in \mathbf{C}$, $b \in \mathbf{Z}_{\geq 0}$.

Thus, the functions $f_{\pm\infty}$ arising in (4) are typical finite functions, and the existence of a meromorphic continuation of $f^\wedge$ to some larger annulus with finitely many poles corresponds to an approximation of f near the boundary of $\mathbf{Z}$ by finite functions.

For f satisfying the asymptotic expansions (4), an application of Exercise 1.4 to the pair $\{f_{+\infty}, f_{-\infty}\}$ yields a function of the shape

$$F(n) := \sum_{k=1}^{K} a_k f(n + c_k)$$

whose fundamental interval contains $(R_{+\infty}, R_{-\infty})$; the meromorphic continuation of $f^\wedge$ may then be given alternatively by the formula

$$f^\wedge(z) := (\sum_{k=1}^{K} a_k z^{c_k})^{-1} F^\wedge(z). \tag{10}$$

Exercise 1.6. Let $f(n) := 1_{n\geq 1}(-1)^{n-1}n$. Show using (10) for suitable F that $f^\wedge(1) = 1/4$.

1.1.4. *Regularized Fourier transform.* If the fundamental interval of f is *empty* but f nevertheless admits asymptotic expansions of the shape (4) for some nonempty interval $(R_{+\infty}, R_{-\infty})$, then one may use either of the identities (7), (10) to *define* a meromorphic function $f^\wedge$ on the annulus $R_{+\infty} < |z| < R_{-\infty}$, whose polar parts may then be determined by summing those as in (5). We call $f^\wedge$ the *regularized Fourier transform* of f.

The regularized Fourier transform still obeys the formal identities listed in (2). Conversely, (even the first of) those identities characterize it the sense that given a nonempty subinterval $I := (R_{+\infty}, R_{-\infty})$ of $(0, \infty)$, the regularized Fourier transform is the unique equivariant linear extension of the Fourier transform, defined initially on the space of functions $\mathbf{Z} \to \mathbf{C}$ whose fundamental interval contains I, to the larger space of functions admitting asymptotic expansions as in (4). Here "equivariant" means "satisfying the first entry of the table (2)." This characterization follows from the second definition (10).

It is useful computationally to observe that any finite function f has zero regularized Fourier transform $f^\wedge \equiv 0$, as follows for instance either from Exercise 1.4 or from the definition (7) and the pair of identities (9).

1.2. **The Fourier transform on R.** For an integrable function $f : \mathbf{R} \to \mathbf{C}$ its Fourier transform is defined for $\xi \in \mathbf{R}$ by the formula

$$f^\wedge(\xi) := \int_{x\in\mathbf{R}} f(x) e(-\xi x)\, dx, \quad e(x) := e^{2\pi i x}. \tag{11}$$

If f is continuous and $f^\wedge$ is integrable, then the Fourier inversion formula

$$f(x) = \int_{\xi\in\mathbf{R}} f^\wedge(\xi) e(\xi x) \tag{12}$$

holds. One has for $\eta \in \mathbf{R}$, $b \in \mathbf{R}^\times$, $c \in \mathbf{R}$, $a_k \in \mathbf{C}$, $c_k \in \mathbf{R}$ the following formal identities:

(13)

$f(x)$	$f^\wedge(\xi)$
$f(x+c)$	$e(\xi c) f^\wedge(z)$
$e(\eta x) f(x)$	$f^\wedge(\xi - \eta)$
$x f(x)$	$-\frac{1}{2\pi i} \frac{d}{d\xi} f^\wedge(\xi)$
$f(bx)$	$\lvert b\rvert^{-1} f^\wedge(b^{-1}\xi)$
$f(-x)$	$f^\wedge(-\xi)$
$\sum_{k=1}^K a_k f(c_k - x)$	$(\sum_{k=1}^K a_k e(\xi c_k)) f^\wedge(-\xi)$
$\frac{d}{dx} f(x)$	$2\pi i \xi f^\wedge(\xi)$
$\int_{t=-\infty}^x f(t)\,dt$	$(2\pi i \xi)^{-1} f^\wedge(\xi)$

Now let $f : \mathbf{R} \to \mathbf{C}$ be any locally integrable function. Much as in §1.1, the definition (11) makes sense for complex arguments $\xi \in \mathbf{C}$ with imaginary parts lying in a certain (possibly empty) maximal open interval, and defines there a holomorphic function of ξ, whose possible meromorphic continuation to larger horizontal strips in the complex plane corresponds to the existence of asymptotic expansions for f near the boundary of $\mathbf{R}$ by finite functions $f_{\pm\infty}$, which now take the shape of finite linear combinations of functions $x \mapsto e^{ax} x^b$ for $a \in \mathbf{C}$, $b \in \mathbf{Z}_{\geq 0}$; one can also define a regularized Fourier transform under which the finite functions have zero image. We will not need to elaborate further on the above points.

1.3. **The Mellin transform on $\mathbf{R}_+^\times$.** More important for our purposes is the transportation of the definition and discussion from §1.2 via the isomorphism

$$\exp : \mathbf{R} \to \mathbf{R}_+^\times$$

to the multiplicative group of positive reals, upon which we now elaborate. Thus, let $f : \mathbf{R}_+^\times \to \mathbf{C}$ be for a locally integrable function. Denote by $(\sigma_\infty, \sigma_0)$ the maximal open subinterval of $(0, \infty)$ with the property that for each σ inside it, the function $\mathbf{R}_+^\times \ni y \mapsto y^{-\sigma} \lvert f(y)\rvert$ is integrable with respect to the measure $d^\times y := y^{-1}\,dy$; call this interval the *fundamental interval* and the set of all s with real parts therein the *fundamental strip*. The *Mellin transform* of f is defined by the formula

$$f^\wedge(s) := \int_{y \in \mathbf{R}_+^\times} f(y) y^{-s}\, d^\times y$$

for s in the fundamental strip, where it converges normally to a holomorphic function that is bounded for $\mathrm{Re}(s)$ in any fixed compact set. It is related to the Fourier transform of the pullback $f \circ \exp$ of f to $\mathbf{R}$ by the formula

$$f^\wedge(\sigma + it) \int_{x \in \mathbf{R}} f(e^x) e^{-\sigma x} e^{-itx}\, dx.$$

If f is continuous, σ is in the fundamental interval, and $y^s \varphi^\wedge(s)$ is absolutely integrable on the line $\mathrm{Re}(s) = \sigma$, then the inversion formula

$$f(y) = \int_{(\sigma)} f^\wedge(s) y^s \frac{ds}{2\pi i} := \int_{t \in \mathbf{R}} f^\wedge(\sigma + it) y^{\sigma + it} \frac{dt}{2\pi} \tag{14}$$

holds. Conversely, if $F(s)$ is a function holomorphic on some strip $\mathrm{Re}(s) \in (a_\infty, a_0)$ that tends to zero uniformly as $\lvert \mathrm{Im}(s)\rvert \to \infty$ and for which $y^s F(s)$ is absolutely integrable on the line $\mathrm{Re}(s) = \sigma$ for some $\sigma \in (a_\infty, a_0)$, then the function

$$f(y) := \int_{(\sigma)} G(s) y^s \, \frac{ds}{2\pi i}$$

satisfies $f^\wedge(s) = F(s)$. We record once again a small table describing how $f^\wedge$ changes when f is modified in certain simple ways (for $a \in \mathbf{C}, b \in \mathbf{R}^\times$, $c \in \mathbf{R}^\times_+$, $a_n \in \mathbf{C}$, $c_n \in \mathbf{R}^\times_+$):

(15)

$f(y)$	$f^\wedge(s)$
$f(cy)$	$c^s f^\wedge(s)$
$y^a f(y)$	$f^\wedge(s-a)$
$\log(y) f(y)$	$-\frac{d}{ds} f^\wedge(s)$
$f(y^b)$	$\lvert b\rvert^{-1} f^\wedge(b^{-1}s)$
$f(1/y)$	$f^\wedge(-s)$
$\sum_{n=1}^N a_n f(c_n/y)$	$(\sum_{n=1}^N a_n c_n^{-s}) f^\wedge(-s)$
$y \frac{d}{dy} f(y)$	$s f^\wedge(s)$
$\int_0^y f(t) \, d^\times t$	$s^{-1} f^\wedge(s)$

One finds as in §1.1 a close relationship between the smoothness of f and the decay of $f^\wedge$, between the decay of f and the smoothness of $f^\wedge$, between bounds on f near the boundary and the domain of holomorphy/definition of $f^\wedge$, and between asymptotic expansions of f near the boundary and possible meromorphic continuations of $f^\wedge$. For instance, set $\Theta := y \frac{d}{dy}$. One verifies that if f is r-times differentiable and $\Theta^r f$ has fundamental strip containing that of f, then $f^\wedge(s) \ll (1 + |s|)^{-r}$ as $|\mathrm{Im}(s)| \to \infty$ for $\mathrm{Re}(s)$ in any fixed compact set; conversely, if for instance the estimate $f^\wedge(s) \ll (1 + |s|)^{-r-1-\delta}$ holds for such s and some fixed $\delta > 0$, then f is r-times differentiable and $\Theta^r f$ has fundamental strip containing that of f.

To give another example, suppose $f^\wedge$ extends to some neighborhood of a closed strip $\Sigma_\infty \leq \mathrm{Re}(s) \leq \Sigma_0$ containing the fundamental strip $\sigma_\infty < \mathrm{Re}(s) < \sigma_0$, has finitely many poles in that region with none on the lines $\mathrm{Re}(s) = \Sigma_\infty, \Sigma_0$, and satisfies $f^\wedge(s) \ll (1+|s|)^{-1-\delta}$ as $|\mathrm{Im}(s)| \to \infty$ for some fixed $\delta > 0$. Then f admits asymptotic expansions

$$f(y) = f_c(y) + O(y^{\Sigma_c}) \text{ as } y \to c \in \{0, \infty\} \tag{16}$$

for some finite functions f_∞, f_0 corresponding to the polar parts of $f^\wedge$, with a polar part

$$c \frac{b!}{(s-a)^{b+1}}$$

in the strip $\Sigma_\infty < \mathrm{Re}(a) \leq \sigma_\infty$ contributing

$$y \mapsto c y^a \log(y)^b \tag{17}$$

towards f_∞ and similarly

$$-c \frac{b!}{(s-a)^{b+1}}$$

in $\sigma_0 \leq \mathrm{Re}(a) < \Sigma_0$ contributing (17) towards f_0. The proof is via (17) and a contour shift: one applies the decay hypotheses on $f^\wedge$ and Cauchy's theorem for the rectangular contours bounding the regions

$$\{s : \mathrm{Re}(s) \in (\Sigma_\infty, \sigma), \mathrm{Im}(s) \in (-R, R)\}$$

and

$$\{s : \mathrm{Re}(s) \in (\sigma, \Sigma_0), \mathrm{Im}(s) \in (-R, R)\}$$

for $R \to \infty$ and $\sigma \in (\sigma_\infty, \sigma_0)$ to establish first the exact identities

$$f(y) = f_c(y) + \int_{(\Sigma_c)} f^\wedge(s) y^s \, \frac{ds}{2\pi i},$$

from which the claimed estimate follows from another application of the decay hypotheses on $f^\wedge$.

Conversely, if f admits such asymptotic expansions then $f^\wedge$ extends meromorphically to $\Sigma_\infty < \mathrm{Re}(s) < \Sigma_0$ with such polar parts; one can prove this either as in §1.1.2 by splitting

$$\int_{y \in \mathbf{R}_+^\times} = \int_{y=0}^{1} + \int_{y=1}^{\infty}$$

or as in §1.1.3 by finding a finite linear combination of translates $F(y) := \sum a_k f(c_k y)$ whose fundamental interval contains $(\Sigma_\infty, \Sigma_0)$ and setting $f^\wedge(s) := (\sum a_k c_k^s)^{-1} F^\wedge(s)$. If one knows in addition to (16) that for each fixed $r \in \mathbf{Z}_{\geq 1}$, the rth derivative of f satisfies the termwise differentiated relation

$$\Theta^r f(y) = \Theta^r f_c(y) + O(y^{\Sigma_c}) \text{ as } y \to c,$$

then $f^\wedge$ *decays rapidly in vertical strips* in the sense that $f^\wedge(s) \ll (1 + |s|)^{-A}$ as $\mathrm{Im}(s) \to \infty$ for each fixed $A \geq 1$ and for $\mathrm{Re}(s)$ in a fixed compact set.

For example, suppose that $f(y) \ll |y|^{-A}$ for each fixed A as $y \to \infty$ and that there exist complex coefficients b_n ($n \in \mathbf{Z}_{\geq 0}$) so that for each $N \in \mathbf{N}$, one has

$$\Theta^r f(y) = \Theta^r \sum_{n<N} b_n y^n + O(y^N)$$

as $y \to 0$. Then $f^\wedge(-s)$ admits a meromorphic continuation to the entire complex plane with polar parts $b_n/(s+n)$ for $n \in \mathbf{Z}_{\geq 0}$ and rapid decay in vertical strips. These hypotheses apply when $f(y) := e^{-y}$ with $b_n := (-1)^n/n!$, whose Mellin transform $f^\wedge(-s) =: \Gamma(s)$ is thus holomorphic on $\mathbf{C}$ away from poles at $s = -n \in \mathbf{Z}_{\leq 0}$ of residue $(-1)^n/n!$ and has rapid decay in vertical strips.

Finally, even if the fundamental strip of f is empty but f admits the asymptotic expansions (16) for some $\Sigma_\infty < \Sigma_0$, one can define (as in §1.1.4) a *regularized Mellin transform* $f^\wedge$ which satisfies and is characterized by the expected formal identities (13). Formally,

$$f^\wedge(s) := \int_1^\infty f(y) y^{-s} \, d^\times y + \int_0^1 f(y) y^{-s} \, d^\times y$$

where the terms on the RHS are interpreted as the meromorphic continuations of the respective integrals from their original domains of definition, which may be non-overlapping. Moreover, $f^\wedge \equiv 0$ for any finite function f.

1.4. **The Mellin transform on $\mathbf{R}^\times$.** The group $\mathbf{R}^\times$ surjects onto $\mathbf{R}_+^\times$ via the absolute value map with compact (indeed, finite) kernel $\{\pm 1\}$, so it admits a Mellin theory much like that of $\mathbf{R}_+^\times$, as follows. Let $\mathfrak{X}(\mathbf{R}^\times)$ denote the character group of $\mathbf{R}$. We may identify it with the set of ordered pairs $(s, a) \in \mathbf{C} \times \{0, 1\}$ by setting $\chi = |.|^s \,\mathrm{sgn}^a$, thus $\chi(y) = |y|^s \,\mathrm{sgn}(y)^a$ for $y \in \mathbf{R}^\times$ with $\mathrm{sgn}(y) := y/|y|$. We write

$$s(\chi) \in \mathbf{C}, \quad a(\chi) \in \{0, 1\}$$

for the coordinates of χ with respect to this identification. The coordinate s gives $\mathfrak{X}(\mathbf{R}^\times)$ the structure of a complex manifold. The real part of χ is defined to be the real number $\mathrm{Re}(\chi)$ with the property that the identity

$$|\chi(y)| = |y|^{\mathrm{Re}(\chi)} \tag{18}$$

holds for all $y \in \mathbf{R}^\times$, thus $\mathrm{Re}(\chi) := \mathrm{Re}(s)$.

Given each $f : \mathbf{R}^\times \to \mathbf{C}$, one defines as before a fundamental interval $(\sigma_\infty, \sigma_0)$ maximal with respect to the property that $\int_{\mathbf{R}^\times} |f(y)||y|^{-\sigma} \, d^\times y < \infty$ for all σ inside it. The fundamental strip is defined to be the set of all χ with $\mathrm{Re}(\chi)$ in the fundamental interval, and the Mellin transform is defined there to be

$$f^\wedge(\chi) := \int_{y \in \mathbf{R}^\times} f(y)\chi^{-1}(y) \, d^\times y. \tag{19}$$

It is holomorphic with respect to the natural complex structure, and satisfies an inversion formula

$$f(y) = \int_{\mathrm{Re}(\chi)=\sigma} f^\wedge(\chi)\chi(y), \tag{20}$$

where $\int_{\mathrm{Re}(\chi)=\sigma}$ denotes an integral over the set denoted $\mathfrak{X}(\mathbf{R}^\times)(\sigma)$ of all χ with $\mathrm{Re}(\chi) = \sigma$ with respect to a suitable measure, provided that this integral converges absolutely; explicitly, the RHS of (20) is defined to be

$$\frac{1}{2} \sum_{a \in \{0,1\}} \int_{\mathrm{Re}(s)=\sigma} f^\wedge(|.|^s \operatorname{sgn}^a)|y|^s \operatorname{sgn}(y)^a \, \frac{ds}{2\pi i}$$

The relation between meromorphic continuation of $f^\wedge$ and asymptotic expansions of f by finite functions near 0 and ∞ (which now take the form $\chi(y)\log(|y|)^b$ for some $\chi \in \mathfrak{X}(\mathbf{R}^\times)$, $b \in \mathbf{Z}_{\geq 0}$), as well as the definition and properties of the regularized Mellin transform, are as in §1.3.

2. Interaction between Fourier and Mellin transforms

2.1. Mellin transform of the restriction of a Schwartz function on R. Recall that the Schwartz space $\mathcal{S}(\mathbf{R})$ consists of functions which together with all fixed derivatives are of rapid decay in the sense that $\sup_{x \in \mathbf{R}} |x|^i |f^{(j)}(x)| < \infty$ for all $i, j \in \mathbf{Z}_{\geq 0}$, and that the Fourier transform maps $\mathcal{S}(\mathbf{R})$ to itself. Given $f \in \mathcal{S}(\mathbf{R})$, one can make sense both of its additive Fourier transform on $\mathbf{R}$ as well as the Mellin transform of its restriction to $\mathbf{R}^\times$. For the sake of disambiguation let us define $\mathcal{F}f \in \mathcal{S}(\mathbf{R})$ and $\mathcal{M}f$, the latter defined on a suitable subset of $\mathfrak{X}(\mathbf{R}^\times)$, by the formulas

$$\mathcal{F}f(\xi) := f^\wedge(\xi)$$

and

$$\mathcal{M}f(\chi) := (f|_{\mathbf{R}^\times})^\wedge(\chi^{-1}),$$

so that

$$\mathcal{F}f(\xi) = \int_{x \in \mathbf{R}} f(x)e(-\xi x) \, dx, \tag{21}$$

$$\mathcal{M}f(\chi) = \int_{y \in \mathbf{R}^\times} f(y)\chi(y) \, d^\times y. \tag{22}$$

When $\chi = |.|^s$ we shall abbreviate

$$\mathcal{M}f(s) := \mathcal{M}f(|.|^s).$$

We shall never use the symbol 1 to denote the trivial character χ in this connection, so the notation $\mathcal{M}f(1) := \mathcal{M}f(|.|^1)$ shall not create any ambiguity.

Let $f \in \mathcal{S}(\mathbf{R})$. Then f decays rapidly at ∞ and admits an asymptotic expansion

$$f(x) = \sum_{n \in \mathbf{Z} : |n| < N} a_n x^n + O(x^N) \tag{23}$$

as $x \to 0$ for each fixed $N \in \mathbf{N}$. For $y \in \mathbf{R}^\times$, we may rewrite (23) in terms of our parametrization of characters $\chi = |.|^s \operatorname{sgn}^a$ of $\mathbf{R}^\times$ as

$$f(x) = \sum_{n \in \mathbf{Z}_{\geq 0} : |n| < N} \sum_{a \in \{0,1\} : a \equiv n(2)} a_n |x|^n \operatorname{sgn}(x)^a.$$

The analysis of §1.3 shows that $\mathcal{M}f$, defined initially in the strip $\operatorname{Re}(\chi) > 0$, extends meromorphically to all of $\mathfrak{X}(\mathbf{R}^\times)$ where it is holomorphic away from poles at $\chi = |.|^{-n} \operatorname{sgn}^a$ for $n \in \mathbf{Z}_{\geq 0}$, $a \in \{0, 1\}$ satisfying $a \equiv n(2)$, where the residue is $2a_n$. In particular, for each $f \in \mathcal{S}(\mathbf{R})$ the ratio

$$\frac{\mathcal{M}f(\chi)}{\zeta_\infty(\chi)} \tag{24}$$

extends to an entire function of χ, where we set

$$\zeta_\infty(|.|^s \operatorname{sgn}^a) := \Gamma_{\mathbf{R}}(s + a).$$

(When $\chi = |.|^s$ we abbreviate

$$\zeta_\infty(s) := \zeta_\infty(|.|^s)$$

with the same conventions as above for $\mathcal{M}f(s)$.) In other words, ζ_∞ is a common divisor of the collection $\{\mathcal{M}f : f \in \mathcal{S}(\mathbf{R})\}$ in the space of meromorphic functions on $\mathfrak{X}(\mathbf{R}^\times)$, where we say that f_1 divides f_2 if f_2/f_1 is entire. In fact, ζ_∞ is a greatest common divisor in that for each $\chi \in \mathfrak{X}(\mathbf{R}^\times)$ there exists $f \in \mathcal{S}(\mathbf{R})$ so that the ratio (24) is nonzero. This can be seen by making for $a \in \{0, 1\}$ the explicit choice

$$f(x) := x^a e^{-\pi x^2}, \tag{25}$$

for which the computation

$$2 \int_{y \in \mathbf{R}^\times_+} y^a e^{-\pi y^2} y^s \, d^\times y = \Gamma_{\mathbf{R}}(s + a)$$

shows that

$$\mathcal{M}f(\chi) = \begin{cases} \zeta_\infty(\chi) & \text{if } a(\chi) = a \\ 0 & \text{if } a(\chi) \neq a, \end{cases} \tag{26}$$

hence the ratio (24) vanishes on one component and is identically 1 on the other component of $\mathfrak{X}(\mathbf{R}^\times)$. Finally, let us observe using the relations

$$\xi^a e(-\xi x) = \left(\frac{-1}{2\pi i} \frac{d}{dx} \right)^a e(-\xi x),$$

$$\int_{\xi \in \mathbf{R}} e^{-\pi \xi^2} e(-\xi x) \, d\xi = e^{-\pi x^2},$$

$$\left(\frac{-1}{2\pi i}\frac{d}{dx}\right)^a e^{-\pi x^2} = i^{-a} e^{-\pi x^2}$$

that the function (25) has Fourier transform

$$\mathcal{F}f(x) = i^{-a} f(x). \tag{27}$$

2.2. **The local functional equation.** It will be of interest to understand how the Fourier transform on $\mathcal{S}(\mathbf{R})$ interacts with the pullback of the Mellin transform, and in particular to compare $\mathcal{M}f$ with $\mathcal{M}\mathcal{F}f$.

Theorem 2.1 (Local functional equation)**.** *For each $f \in \mathcal{S}(\mathbf{R})$, we have the equality of entire functions*

$$\frac{\mathcal{M}\mathcal{F}f(|.|\chi^{-1})}{\zeta_\infty(|.|\chi^{-1})} = \varepsilon_\infty(\chi)\frac{\mathcal{M}f(\chi)}{\zeta_\infty(\chi)} \tag{28}$$

where ε_∞ is the function on $\mathfrak{X}(\mathbf{R}^\times)$ given by

$$\varepsilon_\infty(|.|^s \operatorname{sgn}^a) := i^{-a}.$$

Note that (28) may be rewritten

$$\mathcal{M}\mathcal{F}f(|.|\chi^{-1}) = \gamma_\infty(\chi)\mathcal{M}f(\chi)$$

with

$$\gamma_\infty(\chi) := \varepsilon_\infty(\chi)\frac{\zeta_\infty(|.|\chi^{-1})}{\zeta_\infty(\chi)}.$$

We will give two proofs, one in each of §2.3 and §2.4.

Remark 2.2. The functions $\varepsilon_\infty, \gamma_\infty$ defined above would be different had we chosen a different normalization for the Fourier transform $\mathcal{F}$.

Exercise 2.3. Let $\chi \in \mathfrak{X}(\mathbf{R}^\times)$. Use the consequence

$$|\Gamma_{\mathbf{R}}(\sigma + it)| \asymp |t|^{(\sigma-1)/2} e^{-\pi|t|} \text{ for } t \gg 1, \sigma \ll 1 \tag{29}$$

of Stirling's formula and the fact that $\Gamma_{\mathbf{R}}(s)$ has no zeros and has poles only for $s \in 2\mathbf{Z}_{\leq 0}$ to show that if $\sigma := \operatorname{Re}(\chi)$ belongs to $[\varepsilon, 1-\varepsilon]$ for some fixed $\varepsilon > 0$, then

$$\gamma_\infty(\chi) \asymp C(\chi)^{1/2-\sigma} \tag{30}$$

where for $\chi = |.|^s \operatorname{sgn}^a$ we define

$$C(\chi) := |1 + \operatorname{Im}(s)|.$$

Show also that (30) holds if $t \gg 1$ and $\sigma \ll 1$.

Exercise 2.4. Let $\varphi \in C_c^\infty(\mathbf{R}^\times)$ be fixed. For $t \in \mathbf{R}^\times$ and $\chi \in \mathfrak{X}(\mathbf{R}^\times)$, define

$$G_\varphi(t,\chi) := \int_{y \in \mathbf{R}^\times} \varphi(y) e(ty) \chi(y) \, d^\times y.$$

Assume that $\operatorname{Re}(\chi) \ll 1$. Show using Theorem 2.1 and Exercise 2.3 that for each fixed $\varepsilon > 0$,

$$G_\varphi(t,\xi) \ll C(\chi)^{-1/2+\varepsilon}.$$

[Hint: after replacing $\varphi(y)$ by $\varphi(y)e^{-\sigma y}$ for a suitable bounded $\sigma \in \mathbf{R}$, we may reduce to the case that $\operatorname{Re}(\chi) = \varepsilon$. Now apply (30) and (28) to $f(x) := \varphi(x)e(tx)$, for which $\mathcal{F}f(x) = \mathcal{F}\varphi(x-t)$ and $\int_{x\in\mathbf{R}} \mathcal{F}\varphi(x)\chi^{-1}(x+t)\,dx \ll 1$.] (One can refine this estimate using the method of stationary phase to an asymptotic formula with main term of magnitude $\ll C(\chi)^{-1/2}$, which is best possible.)

2.3. **Proof 1: The Mellin transform of Parseval's formula.** Note first using (26) and (27) that the function f defined in (25) satisfies (28). On the other hand, for each $\chi \in \mathfrak{X}(\mathbf{R}^\times)$ there exists an f as in (25) not vanishing identically at χ. In this way, the proof of (28) for general f reduces to showing that

$$\mathcal{M}f_1(\chi)\mathcal{M}\mathcal{F}f_2(|.|\chi^{-1}) = \mathcal{M}f_2(\chi)\mathcal{M}\mathcal{F}f_1(|.|\chi^{-1}) \tag{31}$$

for all $f_1, f_2 \in \mathcal{S}(\mathbf{R})$. This is basically the Mellin transform of the consequence implied by the Parseval relation on the matrix coefficients

$$\int_{x\in\mathbf{R}} f_1(x)f_2(yx)\,dx$$

for the action of $y \in \mathbf{R}^\times$ on $\mathcal{S}(\mathbf{R})$, as follows: By the change of variables

$$y_1 := x,$$
$$y_2 := yx$$

for which

$$dx = |y_1|\,d^\times y_1, \quad d^\times y = d^\times y_2, \quad y = y_2/y_1,$$

one has for χ with $\mathrm{Re}(\chi) \in (0,1)$ that

$$\int_{y\in\mathbf{R}^\times}\left(\int_{x\in\mathbf{R}} f_1(x)f_2(yx)\,dx\right)\chi(y)\,d^\times y = \mathcal{M}f_1(|.|\chi^{-1})\mathcal{M}f_2(\chi). \tag{32}$$

Therefore (31) is the Mellin transform of the identity

$$\int_{x\in\mathbf{R}} \mathcal{F}f_1(x)f_2(yx)\,dx = \int_{x\in\mathbf{R}} \mathcal{F}f_2(x)f_1(yx)\,dx, \tag{33}$$

to whose proof we now turn. Note by the change of variables $x \mapsto x/y$ with $dx \mapsto |y|^{-1}\,dx$ that

$$\int_{x\in\mathbf{R}} f_1(x)f_2(yx)\,dx = |y|^{-1}\int_{x\in\mathbf{R}} f_2(x)f_1(y^{-1}x)\,dx \tag{34}$$

and by Parseval's formula

$$\int_{x\in\mathbf{R}} \mathcal{F}f_1(x)f_2(x)\,dx = \int_{x\in\mathbf{R}} f_1(x)\mathcal{F}f_2(x)\,dx$$

and the identity $(\mathcal{F}f(y\cdot))(x) = |y|^{-1}\mathcal{F}(y^{-1}x)$ as in table (13) that

$$\int_{x\in\mathbf{R}} \mathcal{F}f_1(x)f_2(yx)\,dx = |y|^{-1}\int_{x\in\mathbf{R}} f_1(x)\mathcal{F}f_2(y^{-1}x)\,dx. \tag{35}$$

Combining (34) with (35) gives (33).

Remark 2.5. One could also (e.g., as is done in Tate's thesis and most treatments) rearrange the argument so that the meromorphic continuation of $\mathcal{M}f(\chi)$ for $f \in \mathcal{S}(\mathbf{R})$, defined initially for $\mathrm{Re}(\chi) > 0$ or for all χ when $f \in \mathcal{S}(\mathbf{R}^\times)$, is deduced from the local functional equation. Also, our explicit computations of $\varepsilon_\infty(\chi)$ are not necessary to deduce a qualitative form of the theorem. For instance, one can deduce the meromorphic continuation of $\gamma_\infty(\chi)$, defined initially for $\mathrm{Re}(\chi) \in (0,1)$, by testing a provisional form of the functional equation against nonzero f first for which $f \in C_c^\infty(\mathbf{R}^\times)$ and then for which $\mathcal{F}f \in C_c^\infty(\mathbf{R}^\times)$. One can also show directly that $\varepsilon_\infty(\chi)$ is a non-vanishing entire function by using the greatest common divisor property of the ratios $\mathcal{M}f(\chi)/\zeta_\infty(\chi)$ noted in §2.1. The above arrangement has been aimed at emphasizing more concretely the relationship between asymptotics of functions and polar behavior of their Mellin transforms (and making clear that this relationship does not depend upon Fourier duality).

2.4. **Proof 2: Uniqueness of $\mathbf{R}^\times$-eigendistributions on $\mathbf{R}$.** We record here another formulation and proof of Theorem 2.1 for which it will be necessary to recall some basic facts about distributions on $\mathbf{R}$. The central result (Theorem 2.10) asserts that the space of tempered distributions on $\mathbf{R}$ transforming under a given character χ of $\mathbf{R}^\times$ is one-dimensional. The local zeta functions $\zeta_\infty(\chi)$ are used to normalize the distinguished basis element $f \mapsto \mathcal{M}f(\chi)/\zeta_\infty(\chi)$ of that space, while the functions $\varepsilon_\infty(\chi)$, $\gamma_\infty(\chi)$ describe how the Fourier transform acts with respect to such bases; see Corollary 2.11.

2.4.1. *Some review on distributions.* The Schwartz space $\mathcal{S}(\mathbf{R})$ on $\mathbf{R}$ is topologized with respect to the seminorms

$$f \mapsto \sup_{x \in \mathbf{R}} |x^i (\frac{d}{dx})^j f(x)|$$

for $i, j \in \mathbf{Z}_{\geq 0}$. This means that a sequence $f_n \in \mathcal{S}(\mathbf{R})$ converges to some $f \in \mathcal{S}(\mathbf{R})$ if for each $i, j \in \mathbf{Z}_{\geq 0}$ one has $\sup_{x \in \mathbf{R}} |x^i (f_n^{(j)}(x) - f^{(j)}(x))| \to 0$.

The space $C_c^\infty(\mathbf{R}^\times)$ of *test functions* on $\mathbf{R}^\times$ is topologized as the inductive limit of the spaces $C_c^\infty(U)$ for U a compact subset of $\mathbf{R}$, which are in turn topologized by the seminorms

$$f \mapsto \sup_{y \in U} |(y \frac{d}{dy})^j f(y)|$$

for $j \in \mathbf{Z}_{\geq 0}$. This means that a sequence $f_n \in C_c^\infty(\mathbf{R}^\times)$ converges to some $f \in C_c^\infty(\mathbf{R}^\times)$ if the f_n are eventually supported on a fixed compact set containing the support of f and if each derivative $f_n^{(j)}$ converges uniformly to f.

Given a topological vector space V, we denote by $V' := \operatorname{Hom}(V, \mathbf{C})$ its continuous dual, that is, the space of continuous linear functionals $V \to \mathbf{C}$. In particular, we define in this way the space $\mathcal{S}(\mathbf{R})'$ of *tempered distributions* on $\mathbf{R}$ and the space $C_c^\infty(\mathbf{R}^\times)'$ of *distributions* on $\mathbf{R}^\times$. To understand the continuity condition a bit more concretely, we remark that a linear map $D : S(\mathbf{R}) \to \mathbf{C}$ belongs to $S(\mathbf{R})'$ if and only if there exists $J \geq 0$ so that

$$|D(f)| \ll \sum_{i,j \in \mathbf{Z} \cap [0,J]} \sup_{x \in \mathbf{R}} |x^i (\frac{d}{dx})^j f(x)|$$

for all $f \in \mathcal{S}(\mathbf{R})$, and that any $D \in \mathcal{S}(\mathbf{R})'$ can be represented as

$$D(f) = \int_{\mathbf{R}} \varphi f^{(n)}$$

for some $n \in \mathbf{Z}_{\geq 0}$ and some continuous function φ satisfying $\varphi(x) \ll (1 + |x|)^{O(1)}$.

The *support* of a distribution D, belonging to either of the above spaces, is defined to be the smallest closed set, call it $\operatorname{supp}(D)$, with the property that $D(f) = 0$ for all f supported outside $\operatorname{supp}(D)$. For example, any element of the space $\mathcal{S}(\mathbf{R})_0'$ of tempered distributions supported at 0 (i.e., with support contained in $\{0\}$) is of the form

$$D = \sum_j c_j (\frac{d}{dx})^j|_{x=0} : f \mapsto \sum_j c_j f^{(j)}(0)$$

for some finitely supported sequence of coefficients $(c_j)_{j \geq 0}$.

There is a continuous inclusion map $C_c^\infty(\mathbf{R}^\times) \hookrightarrow \mathcal{S}(\mathbf{R})$ given by extending an element $f \in C_c^\infty(\mathbf{R}^\times)$ by zero at the origin. By duality, one obtains a restriction map $\mathcal{S}(\mathbf{R})' \to C_c^\infty(\mathbf{R}^\times)'$ with kernel $\mathcal{S}(\mathbf{R})_0'$. Thus one has a short exact sequence

$$0 \to \mathcal{S}(\mathbf{R})_0' \to \mathcal{S}(\mathbf{R})' \to C_c^\infty(\mathbf{R}^\times)' \tag{36}$$

reflecting the geometry of the closed subset $\{0\}$ of $\mathbf{R}$ and its complement $\mathbf{R} - \{0\} = \mathbf{R}^\times$.

The Fourier transform $\mathcal{F}D$ of some tempered distribution $D \in \mathcal{S}(\mathbf{R})'$ is defined by the relation $\mathcal{F}D(f) := D(\mathcal{F}f)$, so that the extension of the Parseval theorem holds.

2.4.2. *Group actions.* The group $\mathbf{R}^\times$ acts on $\mathbf{R}$ with the two orbits $\{0\}, \mathbf{R}^\times$ and acts transitively on $\mathbf{R}^\times$. This induces an action of $t \in \mathbf{R}^\times$ on functions $f : \mathbf{R} \to \mathbf{C}$ or $f : \mathbf{R}^\times \to \mathbf{C}$, denoted by juxtaposition and given by the formula

$$tf(x) := f(xt),$$

and hence an action on distributions D in either of the spaces defined above, given by

$$tD(f) := D(t^{-1}f).$$

For a character $\chi \in \mathfrak{X}(\mathbf{R}^\times)$ and any vector space V on which χ acts, we now define

$$V(\chi) := \{v \in V : tv = \chi(t)v \text{ for } t \in \mathbf{R}^\times\}$$

to be the space of χ-*equivariant* vectors in V. In particular, we define in this way the space

$$P(\chi) := \mathcal{S}(\mathbf{R})'(\chi)$$

χ-equivariant tempered distributions on $\mathbf{R}$, the subspace

$$P_0(\chi) := \mathcal{S}(\mathbf{R})_0'(\chi)$$

of χ-equivariant tempered distributions on $\mathbf{R}$ supported at 0, and the space

$$P_1(\chi) := C_c^\infty(\mathbf{R}^\times)'(\chi)$$

of χ-equivariant distributions on $\mathbf{R}^\times$. From (36) we obtain for each χ the exact sequence

$$0 \to P_0(\chi) \to P(\chi) \to P_1(\chi). \tag{37}$$

2.4.3. *Results.*

Example 2.6. For $\mathrm{Re}(\chi) > 0$, the absolutely convergent integral

$$D_\chi(f) := \mathcal{M}f(\chi) = \int_{y \in \mathbf{R}^\times} f(y)\chi(y)\, d^\times y \tag{38}$$

defines a χ-equivariant tempered distribution $D_\chi \in P(\chi)$.

Lemma 2.7. $P_0(\chi) = 0$ *for* $\mathrm{Re}(\chi) > 0$.

Proof. It follows from the discussion of §2.4.1 that $P_0(\chi)$ is the one-dimensional space spanned by $f \mapsto f^{(n)}$ when χ is of the form $|.|^{-n}\,\mathrm{sgn}^a$ for some $n \in \mathbf{Z}_{\geq 0}$ and $a \in \{0, 1\}$ with $a \equiv n\ (2)$, and that otherwise $P_0(\chi) = 0$. □

Lemma 2.8. $\dim P_1(\chi) = 1$ *for each* χ.

Proof. For each χ, denote by μ_χ the measure on $\mathbf{R}^\times$ given for $f \in C_c(\mathbf{R}^\times)$ by the absolutely convergent integral formula

$$\mu_\chi(f) := \int_{y \in \mathbf{R}^\times} f(y)\, d^\times y.$$

The change of variables

$$\mu_\chi(t^{-1} f) = \int_{y \in \mathbf{R}^\times} f(y/t)\chi(y)\, d^\times y = \chi(t)\mu_\chi(f)$$

reveals that $\mu_\chi \in P_1(\chi)$. Now let $D \in P_1(\chi)$ be arbitrary. Then for each $f, \alpha \in C_c^\infty(\mathbf{R}^\times)$ one has

$$D(f * \alpha) = \mu_\chi(f) D(\alpha). \tag{39}$$

To verify the desired conclusion, it suffices to show for each $f_1, f_2 \in C_c^\infty(\mathbf{R}^\times)$ that

$$D(f_1)\mu_\chi(f_2) = D(f_2)\mu_\chi(f_1).$$

After choosing an approximate identity $\alpha \in C_c^\infty(\mathbf{R}^\times)$ for which $f_i * \alpha$ is sufficiently close to f_i in the test functions topology for each $i = 1, 2$ and invoking the continuity of D, we reduce to verifying that

$$D(f_1 * \alpha)\mu_\chi(f_2) = D(f_2 * \alpha)\mu_\chi(f_1),$$

which is a consequence of (39) applied to $f = f_1, f_2$. □

Lemma 2.9. *The Fourier transform $\mathcal{F}$ induces an isomorphism $P(|.|\chi^{-1}) \cong P(\chi)$ for each χ.*

Proof. Indeed, it follows from the Fourier identity $\mathcal{F}(t^{-1} f) = |t|\mathcal{F}(tf)$ that for $D \in P(\chi)$ and $f \in \mathcal{S}(\mathbf{R})$ one has

$$t\mathcal{F}D(f) = \mathcal{F}D(t^{-1}f) = D(\mathcal{F}t^{-1}f) = |t|D(t\mathcal{F}f) = |.|\chi^{-1}(t)\mathcal{F}D(f),$$

whence that the Fourier transform $\mathcal{F}$ maps $P(\chi)$ into $P(|.|\chi^{-1})$ and vice-versa. □

Theorem 2.10. $\dim P(\chi) = 1$ *for each* χ.

Proof. By the Fourier duality isomorphism given in Lemma 2.9, it suffices to verify the desired conclusion when $\mathrm{Re}(\chi) > 0$. In that case, the exact sequence (36) and the determination of $P_0(\chi)$ in Lemma 2.7 implies that $P(\chi)$ injects into $P_1(\chi)$. By Lemma 2.8, we deduce that $\dim P(\chi) \leq 1$. Example 2.6 now furnishes the equality $\dim P(\chi) = 1$. □

Corollary 2.11. *For* $\mathrm{Re}(\chi) > 0$, *define*

$$D_\chi^* := \frac{D_\chi}{\zeta_\infty(\chi)}. \tag{40}$$

For each $f \in \mathcal{S}(\mathbf{R})$, the map $\chi \mapsto D_\chi^*(f)$ admits an analytic continuation to all $\chi \in \mathfrak{X}(\mathbf{R}^\times)$. The distribution D_χ^* obtained in this way defines a basis element of $P(\chi)$ for each χ. The Fourier transform acts on such basis elements by the formula

$$\mathcal{F}D_{|.|\chi^{-1}}^* = \varepsilon_\infty(\chi) D_\chi^*$$

with $\varepsilon_\infty(\chi)$ as in §2.

Proof. From Theorem 2.10 it follows that for $\mathrm{Re}(\chi) \in (0,1)$ one has

$$\mathcal{F}D_{|.|\chi^{-1}} = \gamma_\infty(\chi) D_\chi \tag{41}$$

for some function $\gamma_\infty(\chi)$. One can explicitly determine $\gamma_\infty(\chi)$ by evaluating (41) on suitable $f \in \mathcal{S}(\mathbf{R})$, such as those given in (25). The rest of the proof is much as in §2. □

2.5. **Dirichlet characters and Gauss sums.** In this section we briefly describe an analogue of the contents of the preceeding sections in which $\mathbf{R}, \mathbf{R}^\times$ are replaced by $\mathbf{Z}/q, (\mathbf{Z}/q)^\times$ for a natural number q.[1]

Recall that a *Dirichlet character* χ of modulus $q \in \mathbf{N}$ is a function $\chi : \mathbf{Z} \to \mathbf{C}$ that factors through the canonical surjection $\mathbf{Z} \to \mathbf{Z}/q$ by a map which we also denote by $\chi : \mathbf{Z}/q \to \mathbf{C}$ that vanishes off the unit group $(\mathbf{Z}/q)^\times \subset \mathbf{Z}/q$ and restricts there to a group homomorphism which we also denote by $\chi : (\mathbf{Z}/q)^\times \to \mathbf{C}^\times$. The character χ is called *primitive* if for each proper divisor q_0 of q, χ does not factor through $\mathbf{Z}/q_0$; in that case, q is called the *conductor* of q. If $q = 1$, then $\chi(n) = 1$ for all n, including $\chi(0) = 1$. On the other hand

$$\chi(0) = 0 \text{ if } q \neq 1. \tag{42}$$

Concretely, a Dirichlet character $\chi : \mathbf{Z} \to \mathbf{C}$ is a function for which $\chi(1) = 1$, $\chi(mn) = \chi(m)\chi(n)$ for all m, n, $\chi(n) = 0$ unless n is coprime to q, and $\chi(n + kq) = \chi(n)$ for each n, k; it is primitive if for each proper divisor q_0 of q there exist n, k so that $\chi(n + kq_0) \neq \chi(n)$.

Definition 2.12. We define a *principal subgroup* H of $(\mathbf{Z}/q)^\times$ to one of the form

$$H = \ker((\mathbf{Z}/q)^\times \to (\mathbf{Z}/q_0)^\times)$$

for some divisor q_0 of q.

Exercise 2.13. Show for a Dirichlet character χ modulo q that the following are equivalent:

(1) χ is primitive.
(2) χ has nontrivial restriction to each nontrivial principal subgroup of $(\mathbf{Z}/q)^\times$. [Hint: two elements n_1, n_2 of $(\mathbf{Z}/q)^\times$ satisfy $n_1 \equiv n_2$ (q_0) if and only if $n_1 n_2^{-1} \equiv 1$ (q_0).]
(3) χ is orthogonal in $L^2(\mathbf{Z}/q)$ to any function $\alpha : \mathbf{Z}/q \to \mathbf{C}$ invariant under some nontrivial principal subgroup H of $(\mathbf{Z}/q)^\times$, that is, for which $\alpha(ah) = \alpha(a)$ for all $h \in H$, $a \in \mathbf{Z}/q$.

The groups

$$\mathbf{Z}/q := \mathbf{Z}/q\mathbf{Z},$$
$$q^{-1}\mathbf{Z}/1 := q^{-1}\mathbf{Z}/\mathbf{Z}$$

are naturally Pontryagin dual to one another, where $\mathbf{Z}/q$ is given the probability Haar measure and $q^{-1}\mathbf{Z}/1$ the counting measure. Given a function $f : \mathbf{Z}/q \to \mathbf{C}$ we define its Fourier transform $\mathcal{F}f : q^{-1}\mathbf{Z}/1 \to \mathbf{C}$ by the formula

$$\mathcal{F}f(\xi) := \frac{1}{q} \sum_{x \in \mathbf{Z}/q} f(x) e(\xi x),$$

[1] One could make the analogy literal, as is done in Tate's thesis and most expositions thereof, by applying the earlier arguments verbatim over the local fields $\mathbf{Q}_p$.

which satisfies the inversion formula

$$f(x) = \sum_{\xi \in q^{-1}\mathbf{Z}/1} \mathcal{F}f(\xi) e(-\xi x).$$

One important consequence of a character χ modulo q being primitive is that its Fourier transform $\mathcal{F}\chi$ is supported on integers n coprime to q, since if $(n,q) \neq 1$ then the function $\alpha(a) := e(na/q)$ is invariant under some nontrivial principal subgroup H of $(\mathbf{Z}/q)^\times$ and hence orthogonal to χ in view of Exercise 2.13. We compute for $n \in (\mathbf{Z}/q)^\times$ via the change of variables $x \mapsto x/n$ that

$$\mathcal{F}\chi(n/q) = \frac{1}{q} \sum_{x \in \mathbf{Z}/q} \chi(x) e(xn/q) = q^{-1/2} \chi^{-1}(n) \varepsilon_q(\chi) \tag{43}$$

with

$$\varepsilon_q(\chi) := \frac{1}{q^{1/2}} \sum_{a \in \mathbf{Z}/q} \chi(a) e_q(a).$$

The quantity $\varepsilon_q(\chi)$ is called a (normalized) *Gauss sum*. Its most important property is:

Theorem 2.14. *Let χ be a primitive Dirichet character of conductor q. Then $|\varepsilon_q(\chi)| = 1$.*

Proof. By the Plancherel theorem

$$\frac{1}{q} \sum_{x \in \mathbf{Z}/q} |f(x)|^2 = \sum_{\xi \in q^{-1}\mathbf{Z}/1} |\mathcal{F}f(\xi)|^2$$

together with the fact that $|\chi| = |\chi^{-1}|$ is the characteristic function of $(\mathbf{Z}/q)^\times$, we obtain using (43) that (with $\varphi(q) := \#(\mathbf{Z}/q)^\times$)

$$\frac{\varphi(q)}{q} = \frac{1}{q} \sum_{x \in \mathbf{Z}/q} |\chi(x)|^2 = \sum_{a \in \mathbf{Z}/q} |\mathcal{F}\chi(a)|^2 = \frac{\varphi(q)}{q} |\varepsilon_q(\chi)|^2,$$

whence $|\varepsilon_q(\chi)| = 1$. ◻

We have the following $\mathbf{Z}/q$ analogue of Theorem 2.1:

Theorem 2.15 (Mod q local functional equation)**.** *Let χ be a primitive Dirichlet character modulo q. Then for each $f : \mathbf{Z}/q \to \mathbf{C}$ one has*

$$\sum_{a \in (\mathbf{Z}/q)^\times} \mathcal{F}f(a/q) \chi(a) = \frac{\varepsilon_q(\chi)}{q^{1/2}} \sum_{a \in (\mathbf{Z}/q)^\times} f(a) \chi^{-1}(a).$$

Proof. The claimed formula is an immediate consequence of Parseval; explicitly, it follows upon expanding the definition of $\mathcal{F}f(a/q)$ and applying the computation (43) of $\mathcal{F}\chi$. ◻

3. Dirichlet series

3.1. **Motivation.** Suppose given a sequence of complex numbers $a_n \in \mathbf{C}$ indexed by natural numbers $n \in \mathbf{N}$ and a function $\varphi : \mathbf{R}^\times_+ \to \mathbf{C}$ for which one is interested in understanding the asymptotics of sums of the shape

$$f(y) := \sum_n a_n \varphi(n/y) \text{ for } y \in \mathbf{R}^\times_+. \tag{44}$$

Assume first only that φ decays rapidly near ∞ and that $a_n \ll n^{O(1)}$. Then for $s \in \mathbf{C}$ with $\mathrm{Re}(s)$ sufficiently large, the double sum/integral

$$f^\wedge(s) = \int_{y \in \mathbf{R}_+^\times} \sum_n a_n \varphi(n/y) y^{-s} \, d^\times y$$

converges absolutely; rearranging and applying the change of variables $y \mapsto ny$, we obtain for $\mathrm{Re}(s)$ large enough that

$$f^\wedge(s) = \varphi^\wedge(-s) D(s)$$

where

$$D(s) := \sum_n \frac{a_n}{n^s}$$

is the *Dirichlet series* attached to the coefficients a_n. The discussion of §1.3 relates the asymptotic behavior of $f(y)$ as $y \to \infty$ to the possible meromorphic continuation and polar behavior of $f^\wedge(s)$ in some region to the left of the half-plane of absolute convergence.

3.2. **Contour shifting in typical cases.** It is often the case in applications that $D(s)$ admits a meromorphic continuation to the complex plane of *moderate growth*, which means that $D(s) \ll (1 + |s|)^{O(1)}$ as $|\mathrm{Im}(s)| \to \infty$ for $\mathrm{Re}(s)$ in any fixed compact set (and perhaps only for s away from some sufficiently sparse collection of poles, e.g., for $\mathrm{Im}(s)$ taking some denumerable set of values tending to $\pm\infty$). Let us assume also that the function φ, as well as each fixed derivative $\Theta^r \varphi$ ($r \in \mathbf{Z}_{\geq 0}$) with $\Theta := y \frac{d}{dy}$, admit asymptotic expansions near 0 of the shape (say)

$$\Theta^r \varphi(y) = \Theta^r \sum_{\beta \in T : \mathrm{Re}(\beta) < A} c_\beta y^\beta + O(y^A) \tag{45}$$

for each fixed A, where T is a subset of $\mathbf{C}$ whose elements have real part tending to $+\infty$ and the c_β are complex coefficients. (Here one could equally well allow terms of the shape $y^\beta \log(y)^\mu$ with $\mu \in \mathbf{Z}_{\geq 0}$.) Then $f^\wedge$ extends to a meromorphic function with simple poles at $s = -\beta$ of residue c_β. Moreover, $f^\wedge$ is of rapid decay in vertical strips away from its poles. The analysis of §1.3 often allows one to deduce that

$$f(y) = R + \int_{(-A)} y^s \varphi^\wedge(-s) D(s) \, \frac{ds}{2\pi i} \tag{46}$$

with

$$R := \sum_{s : \mathrm{Re}(s) > -A} \mathrm{Res} \, y^s \varphi^\wedge(-s) D(s) \tag{47}$$

provided that the contour $\mathrm{Re}(s) = -A$ contains no poles. The integral is trivially estimated by $O(y^{-A})$ if the series $D(s)$ is regarded as fixed.

3.3. **Functional equations.** It may happen that the series $D(s)$ satisfies some sort of duality relating large positive and large negative values of s; for instance, it might be the case that

$$D(s) = \gamma(s) \widetilde{D}(-s)$$

where $\gamma(s)$ is some reasonable meromorphic function and

$$\widetilde{D}(s) := \sum \frac{b_n}{n^s}$$

is some Dirichlet series that converges absolutely for $\mathrm{Re}(s)$ sufficiently large. One may then rewrite the "remainder term" in (46) by taking A sufficiently large and opening the series defining $\widetilde{D}(s)$, giving

$$f(y) = R + \widetilde{f}(1/y) \tag{48}$$

where

$$\widetilde{f}(y) := \sum_n b_n \widetilde{\varphi}(n/y) \tag{49}$$

with

$$\widetilde{\varphi}(y) := \int_{(-A)} y^s \varphi^\wedge(-s)\gamma(s) \, \frac{ds}{2\pi i}. \tag{50}$$

The asymptotic properties of $\widetilde{\varphi}$ may now be studied in terms of the polar behavior of $\varphi^\wedge$ and γ by contour shifting as in §1.3. In particular, if $\varphi^\wedge(-s)\gamma(s)$ has no poles for $\mathrm{Re}(s) < -A$, then it is often the case that $\widetilde{\varphi}$ is roughly of the same shape as φ in the sense that it decays rapidly at ∞ and admits a nice asymptotic expansion near 0. One refers to $\widetilde{f}(1/y)$ as the *dual sum* to $f(y)$. An exact identity like (48), especially in examples as discussed below where R consists of a single term, is generally referred to as an *approximate functional equation*.

3.4. **Using sums to study values of Dirichlet series.** The above discussion has been from the perspective of studying a weighted sum $f(y)$ as in (44) in terms of a polar term R as in (47) and a dual sum $\widetilde{f}(1/y)$. One can also reverse this analysis to study the polar term R in terms of the sum $f(y)$ and its dual $\widetilde{f}(1/y)$. For instance, suppose that the set T of exponents in the asymptotic expansion (45) of φ and its derivatives near zero is the singleton set $T = \{-w\}$ for some $w \in \mathbf{C}$ and that $c_w = 1$, so that $\varphi(y)$ looks very much like y^{-w} for $y \approx 0$; for instance, we might consider the case that $\varphi(y) = y^{-w}$ holds identically for y small enough. Assume also that $D(s)$ is regular at $s = w$. Then $\varphi^\wedge(-s)$ is holomorphic away from its simple pole at $s = w$ of residue 1, so for A sufficiently large one has

$$R = y^w D(w).$$

One can now use the identity (48) to study $D(w)$ in terms of smoothly weighted sums of the coefficients a_n and b_n. This approach is particularly applicable when the series $D(s)$ does not converge at w.

3.5. **Questions of uniformity; analytic conductor.** We have tacitly assumed so far that the coefficients a_n are fixed and the parameter y is tending to ∞. In applications demanded by analytic number theory, it is crucial to allow the a_n and y vary simultaneously. A typical example, relevant for the problem of studying the least quadratic non-residue $n(p) := \min\{a \in \mathbf{N} : (a|p) = -1\}$ modulo a large prime $p \to \infty$, is when $a_n = (n|p)$ is the quadratic residue symbol and $y := p^\alpha$ for some fixed $\alpha \in (0,1)$, say. In that case, establishing $f(y) = o(y)$ for some fixed $\varphi \in C_c^\infty(\mathbf{R}_+^\times)$ implies the upper bound[2] $n(p) \ll p^\alpha$, as otherwise one would have $a_n = 1$ whenever $n/y \in \mathrm{supp}(\varphi)$ and so $f(y) \geq \sum_n \varphi(n/y) \gg y$.

[2]One can immediately improve this estimate further by a combinatorial argument exploiting that $a_n = 1$ for $n \leq x$ implies $a_m = 1$ for each m which is a product of natural numbers $n \leq x$, but describing that refinement would take us too far afield.

Regarding this question of uniformity, we content ourselves for now with two points. First of all, the "error term" in (46) cannot be estimated uniformly by $O(y^{-A})$ if the series $D(s)$ is allowed to vary. This motivates the problem of understanding the true magnitude of a Dirichlet series $D(s)$, particularly in regions where the series representation does not converge absolutely. The method described above in §3.4 is often a first step towards that end.

Secondly, the integral transform $\widetilde{\varphi}$ of φ as defined in (50) may involve a varying function $\gamma(s)$. It is often the case that there exists a parameter $C \geq 1$, called the *analytic conductor* of the sum $f(y)$, with the property that the normalized function

$$C^s \gamma(s)$$

may be regarded as essentially fixed (in the sense of magnitude, growth, polar behavior, etc.) for the purposes of one's analysis.

For example, if

$$\gamma(s) = p^{-s} \frac{\Gamma_{\mathbf{R}}(1/2 - s)}{\Gamma_{\mathbf{R}}(1/2 + s)} \tag{51}$$

as happens when $a_n = (n|p) n^{-1/2}$, then it is of course reasonable to take $C := p$. If instead

$$\gamma(s) = \frac{\Gamma_{\mathbf{R}}(1/2 - (s + iT))}{\Gamma_{\mathbf{R}}(1/2 + (s + iT))} \tag{52}$$

as happens when $a_n = n^{-(1/2 + iT)}$, then the consequence $\gamma(\sigma + it) \asymp |t + T|^{-\sigma}$ of Stirling's formula $\Gamma_{\mathbf{R}}(\sigma + it) \asymp t^{(\sigma - 1)/2} e^{-\pi |t|}$ (with both estimates uniform away from poles and for σ in a fixed compact set) suggests taking $C \asymp 1 + |T|$. In the hybrid case

$$\gamma(s) = p^{-s} \frac{\Gamma_{\mathbf{R}}(1/2 - (s + iT))}{\Gamma_{\mathbf{R}}(1/2 + (s + iT))} \tag{53}$$

arising when $a_n = (n|p) n^{-(1/2 + iT)}$, one might take $C \asymp p(1 + |T|)$. We refer to Exercise 4.11 below for some precise justification of such choices.

It is then convenient to replace the definition (50) with

$$\widetilde{\varphi}(y) := \int_{(-A)} y^s \varphi^\wedge(-s) C^s \gamma(s) \, \frac{ds}{2\pi i} \tag{54}$$

so that $\widetilde{\varphi}$ may be regarded as an essentially fixed function whenever φ is fixed. With the same definition (49) of $\widetilde{f}$, the duality (48) now reads

$$f(y) = R + \widetilde{f}(C/y). \tag{55}$$

Thus the analytic conductor is roughly the product of the effective length of a sum $f(y)$ and that of its dual sum $\widetilde{f}(C/y)$.

When using (55) for the purpose of studying the polar term R, it is reasonable to choose $y := C^{1/2}$ so that the two sums $f(y)$ and $\widetilde{f}(C/y)$ have roughly the same effective length. Thus the polar term R, and hence the values $D(w)$ as in §3.4, may typically be expressed in terms of smoothly weighted sums of the coefficients a_n, b_n of effective length $n \ll \sqrt{C}$.

3.6. **Some general facts.** We record here some general facts about a Dirichlet series $D(s) = \sum_{n \in \mathbf{N}} a_n n^{-s}$. I didn't cover these in lecture, and don't foresee that they'll be of much or any use in the course, but figured it wouldn't hurt to include them here for the sake of completeness.

Theorem 3.1. *There exist $\sigma_c, \sigma_a \in [-\infty, +\infty]$ with $\sigma_c \leq \sigma_a$, called respectively the* abscissa of convergence *and* abscissa of absolute convergence *of $D(s)$, so that for $s = \sigma + it$, the series $D(s)$ converges if $\sigma > \sigma_c$, diverges if $\sigma < \sigma_c$, converges absolutely if $\sigma > \sigma_a$, and diverges absolutely if $\sigma < \sigma_a$. $D(s)$ is holomorphic on the half-plane $\mathrm{Re}(s) > \sigma_c$ where it is represented by its series definition and has derivatives given by termwise differentiation of the series. If the coefficients a_n are nonnegative, then $\sigma_a = \sigma_c$; if moreover σ_a is finite, then $D(s)$ has a singularity there.*

Remark 3.2. In general, one may have $\sigma_c < \sigma_a$, and $D(s)$ may or may not converge absolutely on the line $\mathrm{Re}(s) = \sigma_a$.

We record only the key lemmas needed to prove Theorem 3.1:

Lemma 3.3 (Partial summation)**.** *If $D(s_0)$ converges, then for each $C \geq 1$ the series $D(s)$ converges uniformly for $\mathrm{Re}(s - s_0) > 0$ and $\frac{|s-s_0|}{\mathrm{Re}(s-s_0)} \leq C$.*

Proof. We may assume $s_0 = 0$ after scaling the a_n by n^{-s_0} if necessary. Set $S(t) := \sum_{n \leq t} a_n$; our hypothesis is then that $S(t) = o(1)$ as $t \to \infty$. Let T_1, T_2 be positive reals tending to ∞, and let $s = \sigma + it$ with $\sigma > 0$ and $s/|\sigma| = O(1)$. Our aim is to show that

$$\sum_{T_1 < n \leq T_2} \frac{a_n}{n^s} = o(1)$$

for such s. By partial summation, the above may be written as

$$\int_{T_1}^{T_1} \frac{dS(t)}{t^s} = \frac{S(T_2)}{T_2^s} - \frac{S(T_1)}{T_1^s} + s \int_{T_1}^{T_2} \frac{S(t)}{t^{s+1}}\, dt.$$

As $S(t) = o(1)$ for each $t \in [T_1, T_2]$, it suffices now to show that

$$|s| \int_{T_1}^{T_2} \frac{dt}{t^{\sigma+1}} = \frac{|s|}{\sigma} \left(\frac{1}{T_1^{\sigma}} - \frac{1}{T_2^{\sigma}} \right) = o(1),$$

which follows directly from our assumption on s. □

Lemma 3.4 (Landau's lemma)**.** *If a Dirichlet series $D(s)$ with nonnegative coefficients has finite abscissa of absolute convergence σ_a, then σ_a is a singularity of $D(s)$.*

Proof. We may assume $\sigma_a = 0$. It then suffices to show that the radius of convergence of the Taylor series for $D(s)$ at 1 is (at most) 1. Thus, suppose otherwise that the series

$$D(1 - z) = \sum_{k \geq 0} \frac{(-1)^k D^{(k)}(1)}{k!} z^k$$

converges absolutely for some $z > 1$. Since each of the quantities

$$(-1)^k D^{(k)}(1) = \sum_{n \geq 1} \frac{\log(n)^k}{k!} \frac{a_n}{n}$$

is nonnegative, we deduce that the double sum

$$\sum_{k\geq 0}\sum_{n\geq 1}\frac{\log(n)^k}{k!}\frac{a_n}{n}z^k$$

converges (absolutely) for some $z > 1$. By Fubini's theorem and the identity

$$\sum_{k\geq 0}\frac{\log(n)^k}{k!}z^k = n^z$$

it follows that

$$\sum_{n\geq 1}\frac{a_n}{n^{1-z}}$$

converges absolutely for some $z > 1$, contradicting our assumption that $\sigma_a = 0$. □

4. Basic properties of degree one L-functions

4.1. **The Riemann zeta function.** The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ by the normally convergent series $\zeta(s) = \sum_{n\in\mathbf{N}} n^{-s}$. As a first step towards studying its basic analytic properties, we establish the following:

Theorem 4.1 (Global functional equation)**.** *For each $f \in \mathcal{S}(\mathbf{R})$ the function*

$$\zeta(s)\mathcal{M}f(s),$$

defined initially for $\mathrm{Re}(s) > 1$, *admits a meromorphic continuation to the complex plane with polar parts*

$$\frac{-f(0)}{s}, \quad \frac{\mathcal{F}f(0)}{s-1} \tag{56}$$

and satisfies the functional equation

$$\zeta(s)\mathcal{M}f(s) = \zeta(1-s)\mathcal{M}\mathcal{F}f(1-s).$$

Proof. The proof that follows is to the Poisson summation formula

$$\mathcal{F}\delta_{\mathbf{Z}} = \delta_{\mathbf{Z}}, \quad \delta_{\mathbf{Z}} := \sum_{n\in\mathbf{Z}}\delta_n \tag{57}$$

as the proof of Theorem 2.1 described in §2 is to the Parseval formula. We consider the matrix coefficient

$$\varphi(y) := \int_{x\in\mathbf{R}}\mathcal{F}f_1(x)f_2(yx)\,dx \tag{58}$$

but with $f_1 := \delta_{\mathbf{Z}} = \mathcal{F}\delta_{\mathbf{Z}} \in \mathcal{S}(\mathbf{R})'$ and $f_2 := f \in \mathcal{S}(\mathbf{R})$ arbitrary, thus

$$\varphi(y) = \sum_{n\in\mathbf{Z}} f(yn). \tag{59}$$

The Poisson summation formula (34), (57) implies that

$$\varphi(y) = |y|^{-1}\sum_{n\in\mathbf{Z}}\mathcal{F}f(y^{-1}n). \tag{60}$$

We see from (59) that

$$\varphi(y) = f(0) + O(y^{-\infty}) \text{ as } y \to \infty$$

and from (60) that

$$\varphi(y) = |y|^{-1}\mathcal{F}f(0) + O(y^{\infty}) \text{ as } y \to 0.$$

By §1.3, 1.4, it follows that φ admits a regularized Mellin transform $\mathcal{M}\varphi(s)$ which is meromorphic on $\mathbf{C}$ with polar parts

$$\frac{-2f(0)}{s}, \quad \frac{2\mathcal{F}f(0)}{s-1}.$$

Since the $n = 0$ terms of either expression for $\varphi(y)$ given in (60) is a finite function of y, they may be discarded for the purpose of computing $\mathcal{M}\varphi(y)$; we obtain in this way using the table

(61)

$f(y)$	$\mathcal{M}f(s)$
$f(yn)$	$\lvert n\rvert^{-s}\mathcal{M}f(s)$
$\lvert y\rvert^{-1}\mathcal{F}f(y^{-1}n)$	$\lvert n\rvert^{-(1-s)}\mathcal{M}\mathcal{F}f(1-s)$

and the summations

$$\sum_{0\neq n\in\mathbf{Z}} |n|^{-s} = 2\zeta(s),$$
$$\sum_{0\neq n\in\mathbf{Z}} |n|^{-(1-s)} = 2\zeta(1-s)$$

that

$$\mathcal{M}\varphi(s) = 2\zeta(s)\mathcal{M}f(s), \quad \mathcal{M}\varphi(s) = 2\zeta(1-s)\mathcal{M}\mathcal{F}f(1-s), \tag{62}$$

with the first identity holding initially for $\mathrm{Re}(s) > 1$, the second for $\mathrm{Re}(s) < 0$, and either giving the desired meromorphic continuation of the RHS. The desired conclusions are now immediate. □

Theorem 4.2. *$\zeta(s)$ admits a meromorphic continuation to the complex plane where it satisfies*

$$\xi(s) := \zeta_\infty(s)\zeta(s) = \xi(1-s). \tag{63}$$

$\zeta(s)$ is holomorphic away from a simple pole at $s = 1$ of residues 1.

Proof. Choose $f \in \mathcal{S}(\mathbf{R})$ so that $\mathcal{M}f(s)$ is not identically zero. Then $1/\mathcal{M}f(s)$ is meromorphic, so Theorem 4.1 gives the meromorphic continuation of ζ. Next, by combining the local functional equation (Theorem 2.1)

$$\frac{\mathcal{M}\mathcal{F}f(1-s)}{\zeta_\infty(1-s)} = \frac{\mathcal{M}f(s)}{\zeta_\infty(s)}$$

(using here that $\varepsilon_\infty(s) = 1$, as followed from the computation of the Fourier transform of the Gaussian in §2.1) with the global functional equation (Theorem 4.1)

$$\zeta(s)\mathcal{M}f(s) = \zeta(1-s)\mathcal{M}\mathcal{F}f(1-s)$$

for such an f, we obtain (63). We rewrite

$$\xi(s)\frac{\mathcal{M}f(s)}{\zeta_\infty(s)} = \zeta(s)\mathcal{M}f(s)$$

and appeal to the known determination in (56) of the polar parts of $\zeta(s)\mathcal{M}f(s)$, together with the fact observed in §2.1 that for each s there exists f so that $\mathcal{M}f(s)/\zeta_\infty(s) \neq 0$, to see that $\xi(s)$ is holomorphic away from simple poles $s = 0, 1$. Since $\zeta_\infty(0) = \infty$, we see that $\zeta(s)$ is holomorphic away from $s = 1$. To show that the residue there is 1, it suffices by Theorem 4.1 to choose f for which the residue of $\zeta(s)\mathcal{M}f(s)$ at $s = 1$ is $\mathcal{F}f(0)$; this follows from the identity $\mathcal{M}f(1) = \mathcal{F}f(0)$ after choosing f for which $\mathcal{M}f(1) \neq 0$. □

Remark 4.3. The proofs of Theorems 4.1 and Theorem 4.2 can be somewhat simplified and shortened by choosing $f(x) := e^{-\pi x^2}$ from the start. Nevertheless, I find the above arrangement more instructive and less magical. The argument may be summarized by looking for what relations are implied by the Parseval and Poisson summation formulas on the matrix coefficients for the action of $\mathbf{R}^\times$ on functions/distributions on $\mathbf{R}$.

Remark 4.4. Formally, if one takes for granted the meromorphic continuation and applies the local functional equation (28) with $f = \mathcal{F}\delta_{\mathbf{Z}}$, whose restriction to (a tempered distribution on) $\mathbf{R}^\times$ has Mellin transform

$$\mathcal{M}\delta_{\mathbf{Z}}(1-s) = \sum_{n \in \mathbf{Z}_{\neq 0}} |n|^{1-s}\frac{1}{|n|} = 2\zeta(s),$$

one obtains

$$\frac{2\zeta(s)}{\zeta_\infty(1-s)} = \frac{\mathcal{M}\mathcal{F}\delta_{\mathbf{Z}}(s)}{\zeta_\infty(s)}.$$

In this way, the functional equation (63) for ζ is visibly equivalent to the identity $\mathcal{M}\mathcal{F}\delta_{\mathbf{Z}}(s) = \mathcal{M}\delta_{\mathbf{Z}}(s)$, hence to the Poisson summation formula

$$\mathcal{F}\delta_{\mathbf{Z}} = \delta_{\mathbf{Z}}. \tag{64}$$

The above proof basically makes this formal argument precise with the auxiliary function f playing a regularizing role.

4.2. **Dirichlet L-functions.** Denote by $\mathcal{S}(\mathbf{R}\times\mathbf{Z}/q)$ the space of functions $f(x_\infty, x_q)$ that are Schwartz in the x_∞-variable for each fixed x_q; in other words, this space is the the linear span of functions of the form $f(x_\infty, x_q) = f_\infty(x_\infty)f_q(x_q)$ for some $f_\infty \in \mathcal{S}(\mathbf{R})$ and $f_q : \mathbf{Z}/q \to \mathbf{C}$; denote such functions (called *pure tensors* in what follows) by $f = f_\infty \otimes f_q$.

Define the space $\mathcal{S}(\mathbf{R} \times q^{-1}\mathbf{Z}/1)$ analogously. One has a Fourier isomorphism

$$\mathcal{F} : \mathcal{S}(\mathbf{R} \times \mathbf{Z}/q) \to \mathcal{S}(\mathbf{R} \times q^{-1}\mathbf{Z}/1)$$

given on pure tensors by $\mathcal{F}(f_\infty \otimes f_q) := \mathcal{F}f_\infty \otimes \mathcal{F}f_q$ with $\mathcal{F}$ as defined in §2.1 and §2.5, or explicitly and in general by

$$\mathcal{F}f(\xi_\infty, \xi_q) := \int_{x_\infty \in \mathbf{R}} \frac{1}{q} \sum_{x_q \in \mathbf{Z}/q} f(x_\infty, x_q) e(-\xi_\infty x_\infty + \xi_q x_q).$$

Given a character χ modulo q, denote by $\chi_\infty \in \mathfrak{X}(\mathbf{R}^\times)$ the finite order character $\chi_\infty : \mathbf{R}^\times \to \{\pm 1\}$ with the property that

$$\chi(n) = \chi_\infty(n)\chi(|n|) \tag{65}$$

for all $n = \pm 1$, hence for all $n \in \mathbf{Z}_{\neq 0}$. Thus $\chi_\infty = \operatorname{sgn}^a$ if $a \in \{0, 1\}$ is chosen so that $\chi(-1) = (-1)^a$. For a complex number s and a locally integrable function $f : \mathbf{R}^\times \times (\mathbf{Z}/q)^\times \to \mathbf{C}$, define the Mellin transform

$$\mathcal{M}f(\chi, s) := \int_{y \in \mathbf{R}^\times} \sum_{a \in (\mathbf{Z}/q)^\times} f(y, a)\chi_\infty(y)|y|^s\chi^{-1}(a)\, d^\times y$$

with the same conventions as in §1.3, 1.4 regarding the domain of definition. In particular, we may restrict an element $f \in \mathcal{S}(\mathbf{R} \times \mathbf{Z}/q)$ to the subset $\mathbf{R}^\times \times (\mathbf{Z}/q)^\times$

and define $\mathcal{M}f(\chi, s)$ in this way, which by §2.1 admits a meromorphic continuation in the s variable to all of $\mathbf{C}$. Similarly, for $f \in \mathcal{S}(\mathbf{R} \times q^{-1}\mathbf{Z}/1)$ define

$$\mathcal{M}f(\chi, s) := \int_{y \in \mathbf{R}^\times} \sum_{a \in \mathbf{Z}/q} f(y, a/q) \chi_\infty(y) |y|^s \chi^{-1}(a) \, d^\times y.$$

Theorem 4.5 (Local functional equation on $\mathbf{R} \times \mathbf{Z}/q$)**.** *Let χ be a primitive Dirichlet character modulo q, $s \in \mathbf{C}$, and $f \in \mathcal{S}(\mathbf{R} \times \mathbf{Z}/q)$. Then*

$$\mathcal{M}\mathcal{F}f(\chi^{-1}, 1-s) = \gamma_\infty(\chi_\infty |.|^s) \frac{\varepsilon_q(\chi)}{q^{1/2}} \mathcal{M}f(\chi, s),$$

where γ_∞ is as in §2.2 and ε_q as in §2.5.

Proof. By linearity, it suffices to consider the case that $f = f_\infty \otimes f_q$ is a pure tensor, in which case the claimed formula is the product of those in Theorem 2.1 and Theorem 2.15. □

Note that there are diagonal maps

$$\mathbf{Z} \hookrightarrow \mathbf{R} \times \mathbf{Z}/q$$

and

$$q^{-1}\mathbf{Z} \hookrightarrow \mathbf{R} \times q^{-1}\mathbf{Z}/1$$

given in either case by $n \mapsto (n, n)$.

Remark 4.6. The key property of the pairing

$$(\xi_\infty, \xi_q, x_\infty, x_q) \mapsto e(-\xi_\infty x_\infty + \xi_q x_q)$$

used in what follows is that it induces dualities between $(\mathbf{R} \times \mathbf{Z}/q)/\mathbf{Z}$ and $q^{-1}\mathbf{Z}$ and also between $(\mathbf{R} \times q^{-1}\mathbf{Z}/1)/q^{-1}\mathbf{Z}$ and $\mathbf{Z}$.

Theorem 4.7 (Poisson summation on $\mathbf{R} \times \mathbf{Z}/q$)**.** *For $f \in \mathcal{S}(\mathbf{R} \times \mathbf{Z}/q)$ one has*

$$\sum_{n \in \mathbf{Z}} f(n, n) = \sum_{n \in q^{-1}\mathbf{Z}} \mathcal{F}f(n, n).$$

Proof. Spelled out concretely for a pure tensor $f = f_\infty \otimes f_q$, the claimed formula reads

$$\sum_{n \in \mathbf{Z}} f_\infty(n) f_q(n) = \frac{1}{q} \sum_{n \in \mathbf{Z}} \mathcal{F}f_\infty(n/q) \sum_{a \in \mathbf{Z}/q} f_q(a) e(an/q).$$

By linearity, it suffices to consider the case that $f_q(n) := 1_{n \equiv a(q)}$ for some $a \in \mathbf{Z}/q$. The above then reads

$$\sum_{n \in \mathbf{Z}} f_\infty(a + qn) = \frac{1}{q} \sum_{n \in \mathbf{Z}} \mathcal{F}f_\infty(n/q) e(an/q)$$

which is a consequence of the ordinary Poisson summation formula $\sum_{\mathbf{Z}} h = \sum_{\mathbf{Z}} \mathcal{F}h$ applied to the Fourier pair

$$h(x) := f_\infty(a + qx),$$

$$\mathcal{F}h(x) = q^{-1} \mathcal{F}f_\infty(x/q) e(ax/q);$$

see table (13) (and also Remark 4.6). □

Now let χ be a primitive Dirichlet character of conductor q. The *Dirichlet L-function* attached to χ is the Dirichlet series

$$L(\chi, s) := \sum_{n \in \mathbf{N}} \frac{\chi(n)}{n^s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$. If $\chi = 1$ is the trivial character modulo $q = 1$, then $L(\chi, s) = \zeta(s)$ is the Riemann zeta function. The basic analytic properties of $\zeta(s)$ were shown to follow in §4.1 from consequences of the standard Poisson formula

$$\sum_{n \in \mathbf{Z}} f(n) = \sum_{n \in \mathbf{Z}} \mathcal{F}f(n), \quad f \in \mathcal{S}(\mathbf{R})$$

on matrix coefficients coming from the action of $\mathbf{R}^\times$ on $\mathbf{R}$. The basic properties of $L(\chi, s)$ follow from a similar study of the action of $\mathbf{R}^\times \times (\mathbf{Z}/q)^\times$ on $\mathbf{R} \times \mathbf{Z}/q$, $\mathbf{R} \times q^{-1}\mathbf{Z}$, with Theorem 4.7 playing the role of the Poisson formula. We first give an analogue of Theorem 4.1:

Theorem 4.8 (Global functional equation for $\mathbf{Z} \hookrightarrow \mathbf{R} \times \mathbf{Z}/q$)**.** *Let χ be a primitive Dirichlet character of conductor $q \neq 1$ and $f \in \mathcal{S}(\mathbf{R} \times \mathbf{Z}/q)$. Then the function*

$$L(\chi, s)\mathcal{M}f(\chi, s),$$

defined initially for $\mathrm{Re}(s) > 1$, extends to an entire function on $\mathbf{C}$ where it satisfies the functional equation

$$L(\chi, s)\mathcal{M}f(\chi, s) = q^{1-s} L(\chi^{-1}, 1-s)\mathcal{M}\mathcal{F}f(\chi^{-1}, 1-s). \tag{66}$$

Proof. We proceed as in the proof of Theorem 4.1 by taking the (regularized) Mellin transform in the variables $(y, a) \in \mathbf{R}^\times \times (\mathbf{Z}/q)^\times$ of the consequence

$$\varphi(y, a) := \sum_{n \in \mathbf{Z}} f(yn, an) = |y|^{-1} \sum_{n \in q^{-1}\mathbf{Z}} \mathcal{F}f(y^{-1}n, a^{-1}n)$$

of the Poisson formula given in Theorem 4.7. Using the table[3]

$$\begin{array}{cc} f(y,a) & \mathcal{M}f(\chi, s) \\ \hline f(yn, an) & \chi_\infty^{-1}(n)|n|^{-s}\chi(n)\mathcal{M}f(\chi, s) \\ |y|^{-1}\mathcal{F}f(y^{-1}n, a^{-1}n) & \chi_\infty(n)|n|^{-(1-s)}\chi^{-1}(qn)\mathcal{M}\mathcal{F}f(\chi^{-1}, 1-s) \end{array} \tag{67}$$

and (65) and the evaluations

$$\sum_{n \in \mathbf{Z}_{\neq 0}} \chi(|n|)|n|^{-s} = 2L(\chi, s),$$

$$\sum_{n \in q^{-1}\mathbf{Z}_{\neq 0}} \chi^{-1}(q|n|)|n|^{-(1-s)} = 2q^{1-s}L(\chi^{-1}, 1-s)$$

valid in their respective half-planes, we deduce

$$\mathcal{M}\varphi(\chi, s) = 2L(\chi, s)\mathcal{M}f(\chi, s)$$

and

$$\mathcal{M}\varphi(\chi, s) = 2q^{1-s}L(\chi^{-1}, 1-s)\mathcal{M}\mathcal{F}f(\chi^{-1}, 1-s).$$

From the known meromorphic continuation of $\mathcal{M}\varphi$ (as in the proof of Theorem §4.1) we now obtain the meromorphic continuation of both sides of (66) and then the identity (66) itself. □

[3]Here we invoke the primitivity of χ to see that if $(n, q) \neq 1$ in the first row or $(qn, q) \neq 1$ in the second row, then the Mellin transform recorded in the right column must vanish.

Remark 4.9. The proof can once again be simplified somewhat by choosing f more intelligently, e.g., $f = f_\infty \otimes \chi$ for f_∞ as in §4.1. The purpose of the above arrangement has been to make the argument seem non-magical.

Theorem 4.10. *Let χ be a primitive Dirichlet character of conductor $q \neq 1$. Then $L(\chi, s)$ extends to an entire function on the complex plane where it satisfies*

$$L(\chi, s) = \gamma(\chi, s) L(\chi^{-1}, 1 - s) \tag{68}$$

with

$$\gamma(\chi, s) := \gamma_\infty(\chi_\infty, s)\varepsilon_q(\chi, s)$$

where

$$\varepsilon_q(\chi, s) := \varepsilon_q(\chi) q^{1/2-s}.$$

Equivalently,

$$\Lambda(\chi, s) := L_\infty(\chi, s) L(\chi, s) = \varepsilon(\chi, s)\Lambda(\overline{\chi}, 1 - s) \tag{69}$$

where

$$L_\infty(\chi, s) := \zeta_\infty(\chi_\infty, s)$$

and

$$\varepsilon(\chi, s) := \varepsilon_\infty(\chi_\infty, s)\varepsilon_q(\chi, s) = i^{-a}\varepsilon_q(\chi) q^{1/2-s}$$

with $a := a(\chi_\infty)$.

Proof. We argue as in the proof of Theorem 4.2 by combining the local functional equation of Theorem 4.5 with the global functional equation of Theorem 4.8 for suitable $f \in \mathcal{S}(\mathbf{R} \times \mathbf{Z}/q)$. □

Exercise 4.11. Let notation and assumptions be as in Theorem 4.10. For $s = \sigma + it \in \mathbf{C}$, define the *analytic conductor*

$$C(\chi, s) := q(1 + |t|).$$

Show for $\sigma \ll 1$ and $t \gg 1$ that

$$\gamma(\chi, s) \asymp C(\chi, s)^{1/2-\sigma}.$$

[Use Exercise 2.3 and Theorem 2.15.]

Exercise 4.12. Let notation and assumptions be as in Exercise 4.11. Let $\varepsilon > 0$ be fixed. Show that

$$L(\chi, s) \ll 1 \text{ for } \sigma \geq 1 + \varepsilon$$

and

$$L(\chi, s) \ll C(\chi, s)^{1/2-\sigma} \text{ for } \sigma \leq -\varepsilon.$$

[Use the series representation and the functional equation.]

Exercise 4.13. Let $\varphi \in \mathcal{S}(\mathbf{R})$ be a Schwartz function and χ a primitive Dirichlet character of conductor q. Suppose that φ has the same parity as χ, i.e., that $\mathbf{R}^\times \ni y \mapsto \chi_\infty(y)\varphi(y)$ is an even function. Then one can dualize the sum

$$f(y) := \sum_{n \in \mathbf{Z}} \chi(n)\varphi(n/y) = \chi(0)\varphi(0) + 2\sum_{n \in \mathbf{N}} \chi(n)\varphi(n/y) \tag{70}$$

in two conceivable ways: either by applying the Poisson summation as in Theorem 4.7 to the middle expression in (70) or by dualizing the rightmost expression in (70) according to the recipe of §3.3 using the functional equation of $L(\chi, s)$ established above. Verify that these two approaches give the same answer.

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