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Solution 5

1. By handout [EE2], there exist $r > 0$ such that for every x and y in X,

$$r\varrho(x,y) \leq \sigma(f(x), f(y)) \leq rD_f\varrho(x,y).$$

Apply the two inequalities to the denominator and numerator of $R_{E,F}(\varrho)$ respectively:

$$R_{E,F}(\varrho)^2 = \frac{\sum_{\{u,v\}\in F}\varrho(u,v)^2}{\sum_{\{u,v\}\in E}\varrho(u,v)^2} \leq \frac{r^{-2}\sum_{\{u,v\}\in F}\sigma(f(u),f(v))^2}{r^{-2}D_f^{-2}\sum_{\{u,v\}\in E}\sigma(f(u),f(v))^2} = D_f^2 R_{E,F}(\sigma)^2.$$

2. We note that the assumption on x means that $\sum_{v:v\neq u} x_v = -x_u$ and can thus calculate

$$\begin{aligned}
\sum_{\{v,u\}\in F}(x_u - x_v)^2 &= \frac{1}{2}\sum_{u\in K_n}\sum_{v:v\neq u}(x_u - x_v)^2 \\
&= \frac{1}{2}\left(\sum_u (n-1)x_u^2 - 2\sum_u (x_u \sum_{v\neq u} x_v) + \sum_v (n-1)x_v^2\right) \\
&= \frac{1}{2}\left((n-1)\|x\|^2 + 2\|x\|^2 + (n-1)\|x\|^2\right) = n\|x\|.
\end{aligned}$$

3. To prove

$$\|(\tilde{A}x^{\|})^{\|}\| \leq \overline{\lambda}_G\|x^{\|}\|$$

we first have to understand the matrix $\tilde{A}$. For the construction of the Zig-Zag product of G and H, we distinguished in the intermediate step between blue and red lines (the blue ones correspond to the original edges of H and the red ones come from the rotation map) and an edge in the final graph would corresponding to a path colored blue-red-blue, where the two blue edges would be chosen uniformly at random inside the corresponding copy of H and the red edge is then chosen deterministically. Taking a blue edge at random in every copy of H individually corresponds to the tensor product $\tilde{B} = I_{n_G} \otimes B$ (B adjacency matrix of H) whereas $\tilde{A}$ is determined by the rotation map $(u,i) \mapsto (v,j)$ where v is such that the copy $H_v = H$ is connected to the ith neighbor of u (ordered in G) and j is such that H_u is the copy of H connected to the j neighbor of v. The normalized adjacency matrix of the Zig-Zag product M becomes

$$M = \tilde{B}\tilde{A}\tilde{B}.$$

We make three observations: First, to project a vector on the Zig-Zag product to the space of vectors being constant on each copy of H, we define a map $C : \mathbb{R}^{|n_G||n_H|} \to \mathbb{R}^{|n_G|}$ by averaging over every H copy $(Cx)_u = \sum_{j \in V_H} x_{(u,j)}$ and tensor with the uniform vector $\mathbb{1}_H/n_H$ so that

$$x^{\|} = Cx \otimes \mathbb{1}_H/n_H.$$

Secondly, we observe that for any basis vector e_u on G, its image Ae_u under A is $C\tilde{A}(e_u \otimes \mathbb{1}_H/n_H)$ (draw the picture!). This equality is of linear character and thus holds for any vector, in particular for Cx so that

$$C\tilde{A}x^{\|} = ACx.$$

At last, by definition $x = \tilde{B}w$ for some vector w orthogonal to the uniform vector on the Zig-Zag product. Writing out the definition of C, this implies that Cx is orthogonal to $\mathbb{1}_G$, and hence

$$\|ACx\| \leq \lambda_G \|Cx\|$$

where λ_G is the second largest eigenvalue in absolute value of the normalized adjacency matrix of G. With these observations we see that

$$\begin{aligned} \|(\tilde{A}x^{\|})^{\|}\|^2 = \|C\tilde{A}x^{\|} \otimes \mathbb{1}_G/d_G\|^2 = \frac{1}{d_G}\|C\tilde{A}x^{\|}\|^2 = \frac{1}{d_G}\|ACx\|^2 \\ \leq \lambda_G^2 \frac{1}{d_G}\|Cx\|^2 = \lambda_G^2\|Cx\|^2\|\mathbb{1}_G/d_G\|^2 = \lambda_G^2\|x^{\|}\|^2. \end{aligned}$$

We used twice the definition of the scalar product of the tensor product to open up the norm of $(\tilde{A}x^{\|})^{\|}$ and $x^{\|}$ respectively.