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Solution 6

1. a) For every vertex in the original digraph G there are k outgoing and k incoming edges. The newly constructed graph $G'(L \bigcup R, A)$ now separates all the incoming and outgoing edges into the two copies of $V = L = R$: For a vertex in L there are exactly k outgoing edges (and none incoming). In particular, for any $S \subset L$, there're exactly $k|S|$ edges leaving S. On the other hand, any vertex in R has exactly k incoming edges, so that $|N(S)| \geq k|S|/k = |S|$.

b) The last observation implies by Hall's theorem that there is a perfect matching M_1. Label all the edges in M_1 by "1" and copy this labeling to the original graph G. Since a matching is vertex disjoint, there will be no two incoming (or outgoing) edges at a fixed vertex both labeled "1". We now replace the graph $G'(L \bigcup R, A)$ by $G^{(2)}(L \bigcup R, A \setminus M_1)$. This is again a bipartite digraph but now with $k-1$ outgoing edges in L (and $k-1$ incoming edges in R) for which we may again apply Hall's theorem to get a a edge subset M_2 that we'll label "2". This procedure can be repeated overall k times and will give a consistent labeling.

2. a) There are p^3-1 triples $(a_1, a_2, a_3) \neq (0,0,0)$ and for each line in $\mathbb{F}_p^3$, there are exactly $p-1$ triples representing the line. Therefore, G_p has $(p^3-1)/(p-1) = p^2+p+1$ vertices.

b) Whenever a prime p can be written as a sum of three squares s_1, s_2, s_3, e.g. $p = 17 = 2^2+2^2+3^2$, the vertex representing the line (s_1, s_2, s_3) has a loop.

c) Since any two vertices a and b represent different lines of $\mathbb{F}_p^3$, the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

has rank 2. Therefore, there is exactly one line that is orthogonal to both the line of a and b.

d) Let a be a vertex representing a line characterised by a triple (a_1, a_2, a_3). There exists p^2-1 non-trivial triples (b_1, b_2, b_3) satisfying $a_1b_1+a_2b_2+a_3b_3 = 0$. Each line orthogonal to the line corresponding to the vertex a can be represented by exactly $p-1$ choices of a triple (b_1, b_2, b_3), hence the degree of a in G is $p+1$.

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e) The diagonal entries of A^2 are equal to the degrees of the corresponding vertices (they count the number of walks from a to a of length two). The part (d) implies that all the diagonal entries of A^2 are equal to $p+1$. The part (c) implies that every non-diagonal entry of A^2 is equal to one.

f) The all ones matrix J of size n has rank 1, therefore J has eigenvalue 0 with multiplicity $n-1$. On the other hand, all ones vector of length n is an eigenvector of J, and the corresponding eigenvalue is n.

g) Since $A^2 = pI + J$ and the matrix pI shifts the eigenvalues by p, the analysis of the eigenvalues of J implies that the eigenvalues of A^2 are $p^2 + 2p + 1 = (p+1)^2$ and $p\ldots$

h) $\ldots$ and, finally, the eigenvalues of A are $p+1$ and $\sqrt{p}$.