Problem set – Week 12

First-order differential equations

1. Solve the following differential equations.

   (a) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

   (b) $\frac{dy}{dx} = \frac{\cos^2 y}{\sin^2 x}$

2. Describe geometrically the set of curves that are orthogonal to the integral curves for the differential equation $ydx = xdy$.

3. Write down a differential equation of the form $y' = f(y)$ with solution

   (a) $y(x) = x^\alpha$

   (b) $y(x) = \ln(x)$

   (c) $y(x) = \tan(x)$

   (d) $y(x) = \arcsin(x)$

4. Solve the differential equation
$$\frac{dy}{dx} = \frac{2 - \sin(x + 2y)}{2\sin(x + 2y)}.$$

5. Let α be a real number and consider the initial value problem (IVP)
$$(*) \qquad \frac{dy}{dx} = y^\alpha, \quad y(0) = 0.$$

   (a) Show that this IVP has no solution if $\alpha = 1$.

   (b) For $\alpha \neq 1$, determine the integral curve for $(*)$.

   (c) Find the condition on α for $(*)$ to have a solution $y(x)$ defined for all $x \geq 0$.

   (d) Give an α for which $(*)$ has two solutions.

6. Find a curve C passing through the point $(3, 2)$ with the property that each point p on C is exactly the midpoint of the tangent line to C at p in the first quadrant.

   (a) Sketch what C should look like.

   (b) Let $p = (x, y)$ be a point on C. Determine the slope of the tangent line to C at p.

   (c) Set up the initial value problem for which C is an integral curve.

   (d) Determine the equation of the curve by solving the IVP from (c).

7. An executive conference room of a corporation contains 125 m^3 of air initially free of carbon monoxide (CO). At time $t = 0$, cigarette smoke containing 4% CO is blown into the room at a rate of $r = .005$ m^3/min. A ceiling fan keeps the air in the room circulating so that it leaves the room at rate r. How long does it take for the concentration of CO in the room to reach .01% ?