Problem set – Week 2

Linear dependence & matrix multiplication

1. A bit of vector algebra.

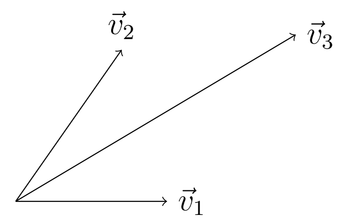
   (a) Find all vectors perpendicular to $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$. Drawing a sketch might help !

   (b) Find all solutions x_1, x_2, x_3 of the equation $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$, where

   $$\vec{b} = \begin{pmatrix} -8 \\ -1 \\ 2 \\ 15 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \\ 5 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 3 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 4 \\ 6 \\ 9 \\ 1 \end{pmatrix}.$$

   (c) Are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ linearly dependent ?

2. Consider the three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in the x-y-plane :



   Are $\vec{v}_1$, $\vec{v}_2$ linearly dependent ? What about $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$? Argue geometrically.

3. Write the system

$$\left| \begin{array}{ccccccc} x & + & 2y & + & 3z & = & 1 \\ 4x & + & 5y & + & 6z & = & 4 \\ 7x & + & 8y & + & 9z & = & 9 \end{array} \right|$$

   in matrix form.

4. If possible, compute the following matrix products.

(a) $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ (b) $\begin{pmatrix}1 & -1\\ -2 & 2\end{pmatrix}\begin{pmatrix}7 & 5\\ 3 & 1\end{pmatrix}$

(c) $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ (d) $\begin{pmatrix}1 & -1\\ 0 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}3 & 2\\ 1 & 0\end{pmatrix}$

(e) $\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ (f) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

(g) $\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 1 & -1 & -2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 1 & 3\end{pmatrix}$ (h) $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

(i) $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}-6 & 8\\ 3 & -4\end{pmatrix}$ (j) $\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 1\\ 1 & 1\end{pmatrix}$

(k) $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}3\\ 2\\ 1\end{pmatrix}$ (l) $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}$

5. Introducing inverses for 2×2 matrices.

   (a) Find all vectors $\vec{x}$ such that $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \qquad \vec{b} = \begin{pmatrix}2\\ 1\end{pmatrix}.$$

   (b) Prove : The 2×2 matrix $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.
   (**Hint :** Consider the cases $a = 0$ and $a \neq 0$ separately.)

   (c) Prove : If A is invertible, then its inverse is given by

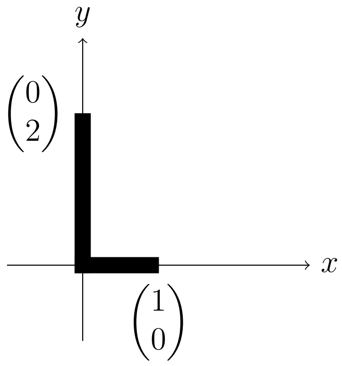
$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}.$$

   (d) Use the formula in (c) to compute $\vec{x}$ in (a).

6. The goal of this exercise is to give a geometric interpretation of the linear transformations defined by the matrices

$$A = \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix} \qquad B = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad C = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Start by showing the effect of these transformations on the letter L :



In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically.