Problem set – Week 3

Matrix multiplication, inverses & calculus warm-up

1. Let $A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 7 \\ 3 & -5 \end{pmatrix}$.

   (a) Compute $5A - B$, A^2, A^{-1}, B^{-1}.

   (b) Show that $(AB)^{-1} = B^{-1}A^{-1}$.

   (c) Check that $A^2 - 5A + I = 0$. Using this equation, find a formula for A^{-1} in terms of A.

2. Find the matrix X such that $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} X = \begin{pmatrix} 28 & 35 & 42 \\ 42 & 53 & 64 \end{pmatrix}$.

3. Compute the first derivative of

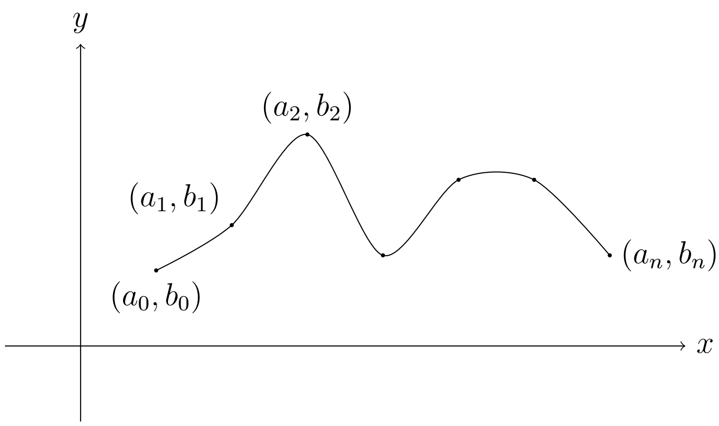
   $$\text{(a) } x^3 e^{-x^3} - x - 3, \qquad \text{(b) } \frac{\log(\sin^2(x))}{\cos(x)}, \qquad \text{(c) } \arctan(\sqrt{x}).$$

   Compute the second derivative of

   $$\text{(d) } \log(\log(x)).$$

4. Find the equation of the line that is normal to the curve $y = (x^4 - 1)^3 \log(x + 1)$ at the origin.

5. You are in charge of designing a roller coaster ride. This simple ride is completely described, as viewed from the side, by the following figure.



You are given the set of points $(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n)$ and you need to connect these in a reasonably smooth way. A method often used is such design problems if that of cubic splines; you want to find polynomials $f_k(t)$ of degree at most 3, to define the shape of the ride between (a_{k-1}, b_{k-1}) and (a_k, b_k) by requiring $f_k(a_{k-1}) = b_{k-1}$ and $f_k(a_k) = b_k$ as well as $f_k'(a_k) = f_{k+1}'(a_k)$, $f_k''(a_k) = f_{k+1}''(a_k)$. Explain the practical significance of these conditions. For the convenience of the riders, it is also required that $f_1'(a_0) = f_n'(a_n) = 0$. Why ?

Show that satisfying all these conditions amounts to solving a linear system !

6. Sketch the curve $y = x^2(x-3)^2$.

7. Let x, y be non-negative integers such that $x+y = 12$. What is the the maximal value of x^2y ?

8. Compute the following integrals

$$\text{(a)} \int \tan(x)\, dx, \qquad \text{(b)} \int \frac{x\, dx}{x^4 + 2x^2 + 2}, \qquad \text{(c)} \int \sin^2(x)\, dx.$$