Problem set – Week 5

Level curves, level surfaces, partial derivatives, vector fields and line integrals

1. For each of the following functions, sketch the surface $z = f(x, y)$ and a typical level curve.

   (a) $f(x,y) = y^2$

   (b) $f(x,y) = 1 - |x| - |y|$

   (c) $f(x,y) = \sqrt{x^2 + y^2 - 4}$

   For each of the following functions, sketch a typical level surface

   (d) $f(x,y,z) = \ln(x^2 + y^2 + z^2)$

   (e) $f(x,y,z) = z$

   (f) $f(x,y,z) = x^2 + y^2$

2. Find the line that is tangent to the intersection of $z = \arctan(xy)$ with the plane $x = 2$ at $(2, 1/2, \pi/4)$.

3. We know that if f_{xy} and f_{yx} exist, then $f_{xy} = f_{yx}$. For the following functions, try to determine what is the fastest way of computing f_{xy} without writing anything down ; should you differentiate in x or in y first ?

   (a) $f(x,y) = x \sin y + e^y$

   (b) $f(x,y) = 1/x$

   (c) $f(x,y) = y + (x/y)$

   (d) $f(x,y) = y + x^2 y + 4y^3 - \ln(y^2 + 1)$

   (e) $f(x,y) = x^2 + 5xy + \sin x + 7e^x$

   (f) $f(x,y) = x \ln xy$

4. Compute the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2r\mathbf{k}$, $t \in [0, 1]$, if the density is $\delta = (3/2)t$.

5. Give a formula $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ for the vector field that points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to $(0, 0)$. Note : The field is not defined at the origin.

6. Determine the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2, 0)$ to itself.

7. A particle is moving along a smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and is always pointing away from the origin. Show that the work done by the force is

$$k\left((b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}\right).$$